Part 3. Spectrum Estimation

3.1 Classic Methods for Spectrum Estimation

Electrical & Computer Engineering

University of Maryland, College Park

Acknowledgment: ENEE630 slides were based on class notes developed by Profs. K.J. Ray Liu and Min Wu. The slides were made by Prof. Min Wu, with updates from Mr. Wei-Hong Chuang. *Contact: minwu@umd.edu*

Logistics

- Last Lecture: lattice predictor
  - correlation properties of error processes
  - joint process estimator in lattice
  - inverse lattice filter structure
- Today:
  - Spectrum estimation: background and classical methods
- Homework set

Summary of Related Readings on Part-II

2.1 Stochastic Processes and modeling

Haykin (4^{th} Ed) 1.1-1.8, 1.12-1.14

Hayes 3.3 – 3.7 (3.5); 4.7

2.2 Wiener filtering

Haykin (4^{th} Ed) Chapter 2

Hayes 7.1, 7.2, 7.3.1

2.3-2.4 Linear prediction and Levinson-Durbin recursion

Haykin (4^{th} Ed) 3.1 – 3.3

Hayes 7.2.2; 5.1; 5.2.1 – 5.2.2, 5.2.4– 5.2.5

2.5 Lattice predictor

Haykin (4^{th} Ed) 3.8 – 3.10

Hayes 6.2; 7.2.4; 6.4.1

Summary of Related Readings on Part-III

Overview Haykins 1.16, 1.10

3.1 Non-parametric method

Hayes 8.1; 8.2 (8.2.3, 8.2.5); 8.3

3.2 Parametric method

Hayes 8.5, 4.7; 8.4

3.3 Frequency estimation

Hayes 8.6

Review

- On DSP and Linear algebra: Hayes 2.2, 2.3
- On probability and parameter estimation: Hayes 3.1 – 3.2

Spectrum Estimation: Background

- Spectral estimation: determine the power distribution in frequency of a random process
  - E.g "Does most of the power of a signal reside at low or high frequencies?" "Are there resonances in the spectrum?"
- Applications:
  - Needs of spectral knowledge in spectrum domain non-causal Wiener filtering, signal detection and tracking, beamforming, etc.
  - Wide use in diverse fields: radar, sonar, speech, biomedicine, geophysics, economics, …
- Estimating p.s.d. of a w.s.s. process is equivalent to estimate autocorrelation at all lags

Spectral Estimation: Challenges

- When a limited amount of observation data are available
  - Can't get *r(k)* for all k and/or may have inaccurate estimate of *r(k)*
  - Scenario-1: transient measurement (earthquake, volcano, …)
  - Scenario-2: constrained to short period to ensure (approx.) stationarity in speech processing

$$\hat{r}(k) = \frac{1}{N-k} \sum_{n=k+1}^{N} u[n] u^*[n-k], \; k = 0,1,\ldots M$$

- Observed data may have been corrupted by noise

Spectral Estimation: Major Approaches

-
  - No assumptions on the underlying model for the data
  - Periodogram and its variations (averaging, smoothing, …)
  - Minimum variance method
-
  - ARMA, AR, MA models
  - Maximum entropy method
-
  - For harmonic processes that consist of a sum of sinusoids or complex-exponentials in noise
-

Spectral Estimation: Major Approaches

- Nonparametric methods
  - No assumptions on the underlying model for the data
  - Periodogram and its variations (averaging, smoothing, …)
  - Minimum variance method
- Parametric methods
  - ARMA, AR, MA models
  - Maximum entropy method
- Frequency estimation (noise subspace methods)
  - For harmonic processes that consist of a sum of sinusoids or complex-exponentials in noise
- High-order statistics

Example of Speech Spectrogram

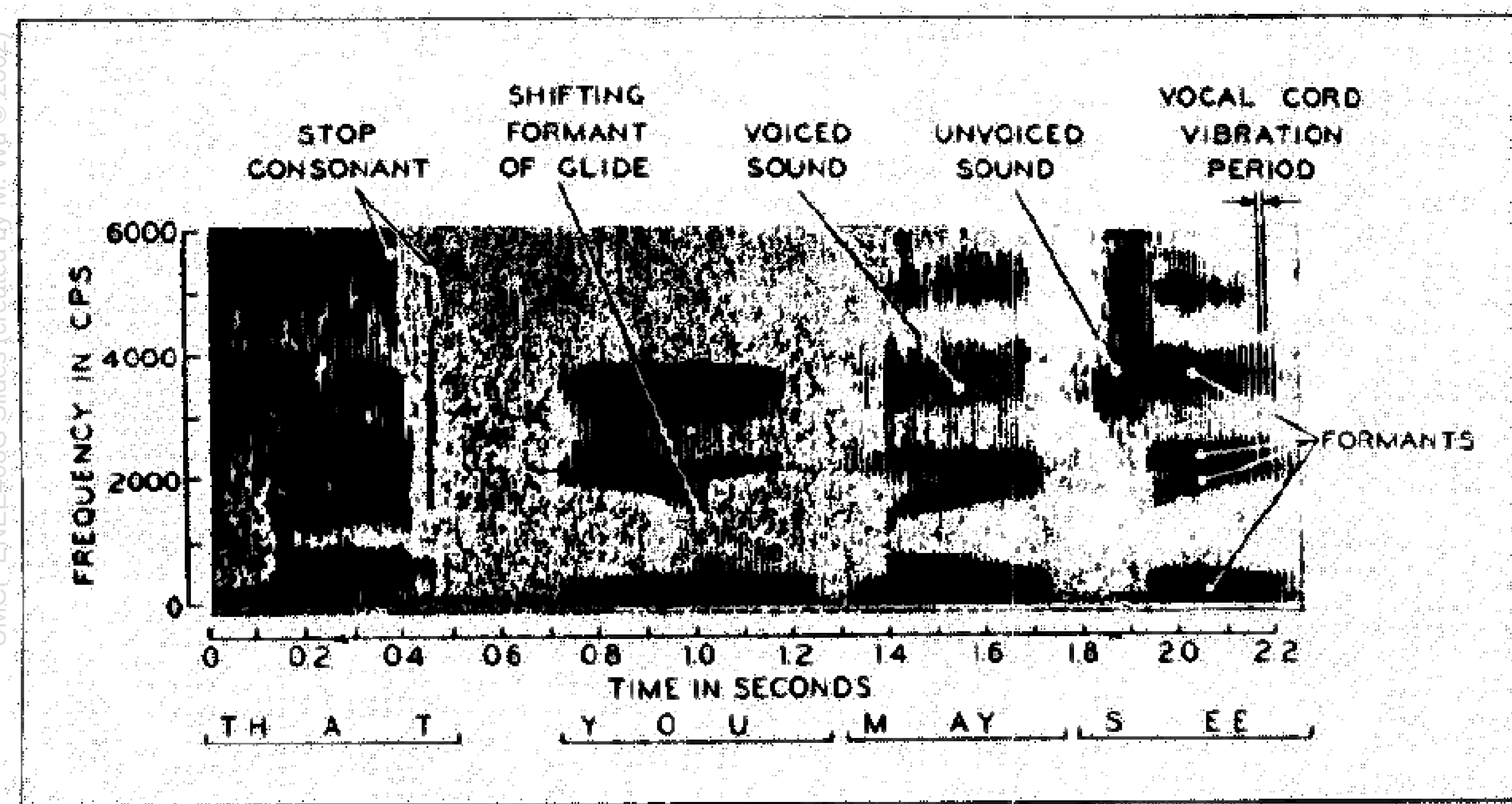
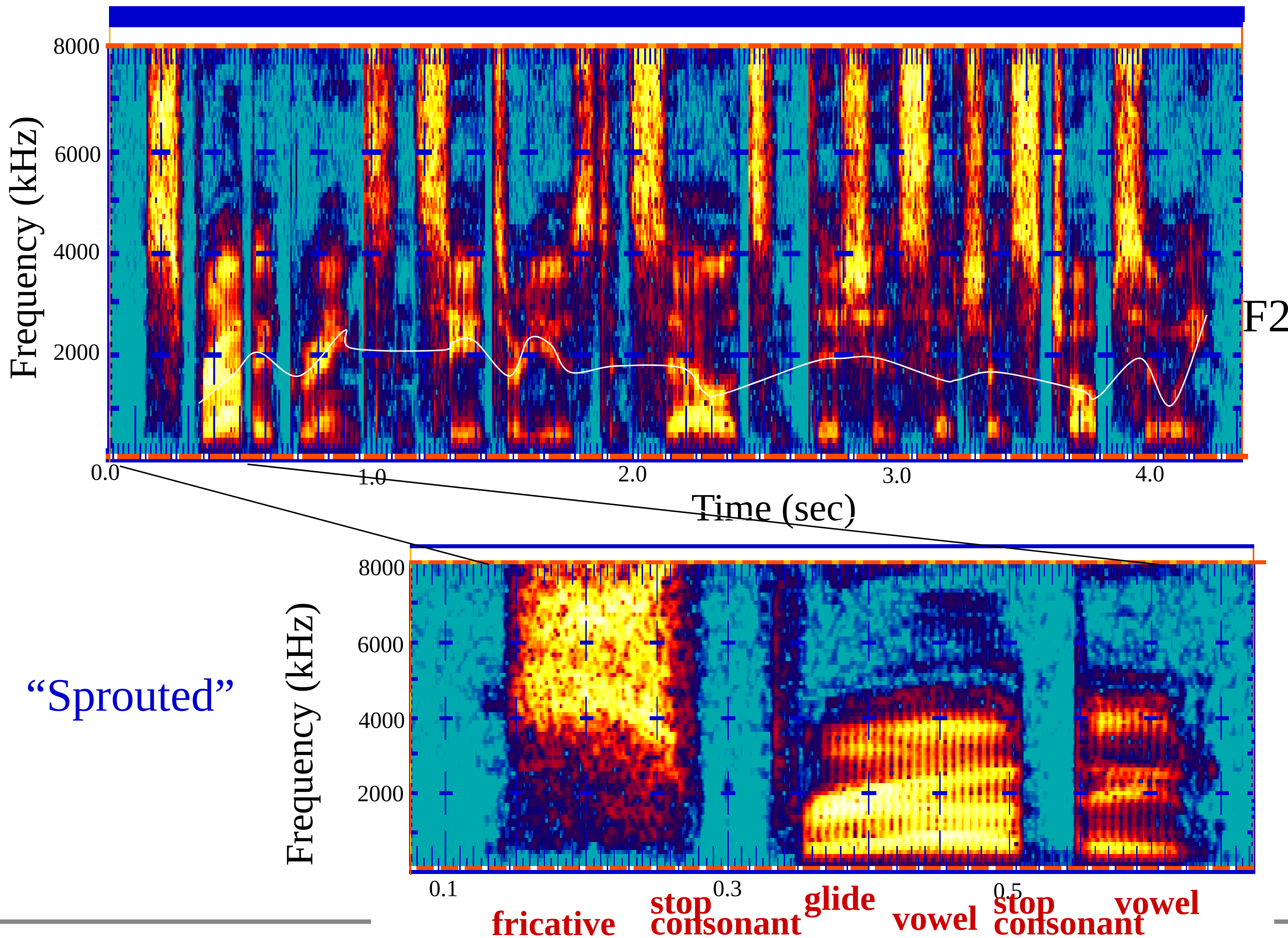


Figure 3 of SPM May'98 Speech Survey

“Sprouted grains and seeds are used in salads and dishes such as chop suey”



Section 3.1 Classical Nonparametric Methods

Recall: given a w.s.s. process {x[n]} with

$$\begin{cases} E[x[n]] = m_x \\ E[x^*[n]x[n+k]] = r(k) \end{cases}$$

The power spectral density (p.s.d.) is defined as

$$-\frac{1}{2} \le f \le \frac{1}{2}$$

$$(\text{or } \omega = 2\pi f : -\pi \le \omega \le \pi)$$

As we can take DTFT on a specific realization of a random process,

What is the relation between the DTFT of a specific signal and the p.s.d. of the random process?

Section 3.1 Classical Nonparametric Methods

Recall: given a w.s.s. process {x[n]} with

$$\begin{cases} E[x[n]] = m_x \\ E[x^*[n]x[n+k]] = r(k) \end{cases}$$

The power spectral density (p.s.d.) is defined as

$$P(f) = \sum_{k=-\infty}^{\infty} r(k) e^{-j2\pi fk} \qquad -\frac{1}{2} \le f \le \frac{1}{2}$$

$$(\text{or } \omega = 2\pi f : -\pi \le \omega \le \pi)$$

As we can take DTFT on a specific realization of a random process,
What is the relation between the DTFT of a specific signal and the p.s.d. of the random process?

Ensemble Average of Squared Fourier Magnitude

- p.s.d. can be related to the ensemble average of the squared Fourier magnitude $|X(\omega)|^2$

$$\text{Consider } P_M(f) \overset{\Delta}{=} \frac{1}{2M+1} \left| \sum_{n=-M}^{M} x[n] e^{-j2\pi f n} \right|^2$$

$$= \frac{1}{2M+1} \sum_{n=-M}^{M} \sum_{m=-M}^{M} x[n] x^*[m] e^{-j2\pi f (n-m)}$$

Ensemble Average of Squared Fourier Magnitude

- p.s.d. can be related to the ensemble average of the squared Fourier magnitude $|X(\omega)|^2$

$$\text{Consider } P_M(f) \stackrel{\Delta}{=} \frac{1}{2M+1} \left| \sum_{n=-M}^{M} x[n] e^{-j2\pi fn} \right|^2$$

$$= \frac{1}{2M+1} \sum_{n=-M}^{M} \sum_{m=-M}^{M} x[n] x^*[m] e^{-j2\pi f(n-m)}$$

i.e., take DTFT on (2M+1) samples and
examine normalized squared magnitude

Note: for each frequency f, $P_M(f)$ is a random variable

Ensemble Average of $P_M(f)$

$$E[P_M(f)] = \frac{1}{2M+1} \sum_{n=-M}^{M} \sum_{m=-M}^{M} r(n-m) e^{-j2\pi f(n-m)}$$

$$= \frac{1}{2M+1} \sum_{k=-2M}^{2M} (2M+1-|k|) r(k) e^{-j2\pi fk}$$

- Now, what if M goes to infinity?

Ensemble Average of $P_M(f)$

$$E[P_M(f)] = \frac{1}{2M+1}\sum_{n=-M}^{M}\sum_{m=-M}^{M} r(n-m)e^{-j2\pi f(n-m)}$$

$$= \frac{1}{2M+1}\sum_{k=-2M}^{2M}(2M+1-|k|)r(k)e^{-j2\pi fk}$$

$$= \sum_{k=-2M}^{2M}\left(1-\frac{|k|}{2M+1}\right)r(k)e^{-j2\pi fk}$$

$$= \sum_{k=-2M}^{2M} r(k)e^{-j2\pi fk} - \frac{1}{2M+1}\sum_{k=-2M}^{2M}|k|r(k)e^{-j2\pi fk}$$

- Now, what if M goes to infinity?

P.S.D. and Ensemble Fourier Magnitude

If the autocorrelation function decays fast enough s.t.

$$\sum_{k=-\infty}^{\infty} |k| r(k) < \infty \quad \text{(i.e., } r(k) \to 0 \text{ rapidly for } k \uparrow\text{)}$$

then $$\lim_{M \to \infty} E[P_M(f)] = \sum_{k=-\infty}^{\infty} r(k) e^{-j2\pi fk} = P(f)$$

p.s.d.

Thus

P.S.D. and Ensemble Fourier Magnitude

If the autocorrelation function decays fast enough s.t.

$$\sum_{k=-\infty}^{\infty} |k| r(k) < \infty \quad (\text{i.e., } r(k) \to 0 \text{ rapidly for } k \uparrow)$$

then

$$\lim_{M \to \infty} E[P_M(f)] = \underbrace{\sum_{k=-\infty}^{\infty} r(k) e^{-j2\pi fk} = P(f)}_{\text{p.s.d.}}$$

Thus

$$P(f) = \lim_{M \to \infty} E\left[\frac{1}{2M+1} \left| \sum_{n=-M}^{M} x[n] e^{-j2\pi fn} \right|^2 \right] \quad (**)$$

3.1.1 Periodogram Spectral Estimator

(1) This estimator is based on (**)

Given an observed data set {x[0], x[1], …, x[N-1]}, the periodogram is defined as

$$\hat{P}_{\mathrm{PER}}(f) = \frac{1}{N}\left|\sum_{n=0}^{N-1} x[n] e^{-j2\pi fn}\right|^2$$

3.1.1 Periodogram Spectral Estimator

(1) This estimator is based on (**)

Given an observed data set {x[0], x[1], …, x[N-1]}, the periodogram is defined as

$$\hat{P}_{\text{PER}}(f) = \frac{1}{N}\left|\sum_{n=0}^{N-1} x[n]e^{-j2\pi fn}\right|^2$$

$$x[n] \xrightarrow[\text{take samples in } n\in[0,N-1]]{} x_N[n] \xrightarrow[\text{DFT}]{} X_N(k) \longrightarrow \underbrace{\frac{1}{N}|X_N(k)|^2}_{\text{periodogram}}$$

An Equivalent Expression of Periodogram

The periodogram estimator can be given in terms of $\hat{r}(k)$

$$\hat{P}_{\text{PER}}(f) = \sum_{k=-(N-1)}^{N-1} \hat{r}(k) e^{-j2\pi fk}$$

where $\hat{r}(k) =$

– The quality of the estimates for the higher lags of r(k) may be poorer since they involve fewer terms of lag products in the averaging operation

Exercise: to show this from the periodogram definition in last page

An Equivalent Expression of Periodogram

The periodogram estimator can be given in terms of $\hat{r}(k)$

$$\hat{P}_{\mathrm{PER}}(f) = \sum_{k=-(N-1)}^{N-1} \hat{r}(k) e^{-j2\pi fk}$$

where $\hat{r}(k) = \frac{1}{N} \sum_{n=0}^{N-1-k} x^*[n]x[n+k]; \hat{r}(-k) = \hat{r}^*(k) \text{ for } k \geq 0$

– The quality of the estimates for the higher lags of r(k) may be poorer since they involve fewer terms of lag products in the averaging operation

Exercise: to show this from the periodogram definition in last page

(2) Filter Bank Interpretation of Periodogram

For a particular frequency of f_0:

$$\hat{P}_{\mathrm{PER}}(f_0) = \frac{1}{N}\left|\sum_{k=0}^{N-1} e^{-j2\pi f_0 k} x[k]\right|^2$$

$$= \left[N \cdot \left|\sum_{k=0}^{N-1} h[n-k]x[k]\right|^2 \right]_{n=0}$$

where

$$h[n] =$$

– Impulse response of the filter h[n]: a windowed version of a complex exponential

(2) Filter Bank Interpretation of Periodogram

For a particular frequency of f_0:

$$\hat{P}_{\text{PER}}(f_0) = \frac{1}{N}\left|\sum_{k=0}^{N-1} e^{-j2\pi f_0 k} x[k]\right|^2$$

$$= \left[N \cdot \left|\sum_{k=0}^{N-1} h[n-k]x[k]\right|^2 \right]_{n=0}$$

where

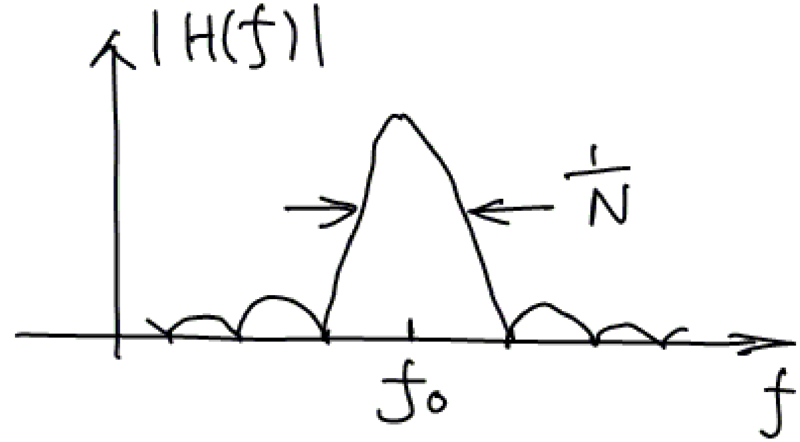
$$h[n] = \begin{cases} \frac{1}{N}\exp(j2\pi f_0 n) & \text{for } n = -(N-1),\ldots,-1,0; \\ 0 & \text{otherwise} \end{cases}$$

– Impulse response of the filter h[n]: a windowed version of a complex exponential

Frequency Response of h[n]

$$H(f) = \underbrace{\frac{\sin N\pi(f - f_0)}{N \sin \pi(f - f_0)}} \exp\left[j(N-1)\pi(f - f_0) \right]$$

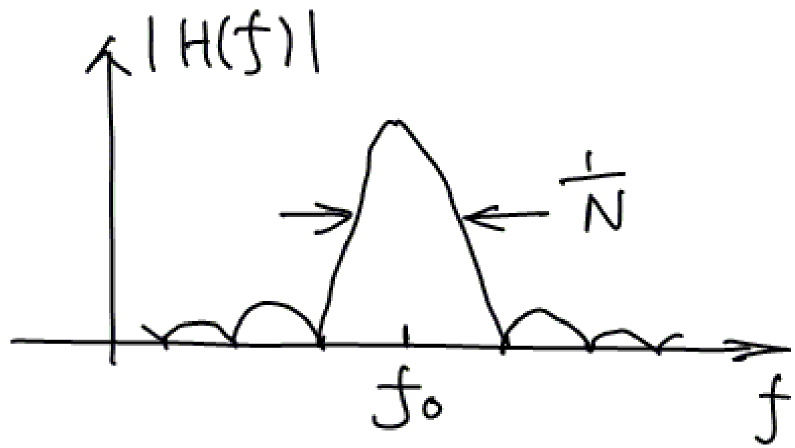
sinc-like function centered at f_0:



Frequency Response of h[n]

$$H(f) = \underbrace{\frac{\sin N\pi(f - f_0)}{N \sin \pi(f - f_0)}}_{\text{sinc-like function centered at } f_{0:}} \exp\left[j(N-1)\pi(f - f_0) \right]$$

- H(f) is a bandpass filter
  - Center frequency is f_0
  - 3dB bandwidth ≈ 1/N



Periodogram: Filter Bank Perspective

- Can view the periodogram as an estimator of power spetrum that has a built-in filterbank
  - The filter bank ~ a set of bandpass filters

$$\hat{P}_{\mathrm{PER}}(f_0) = \left[N \cdot \left| \sum_{k=0}^{N-1} h[n-k]x[k] \right|^2 \right]_{n=0}$$

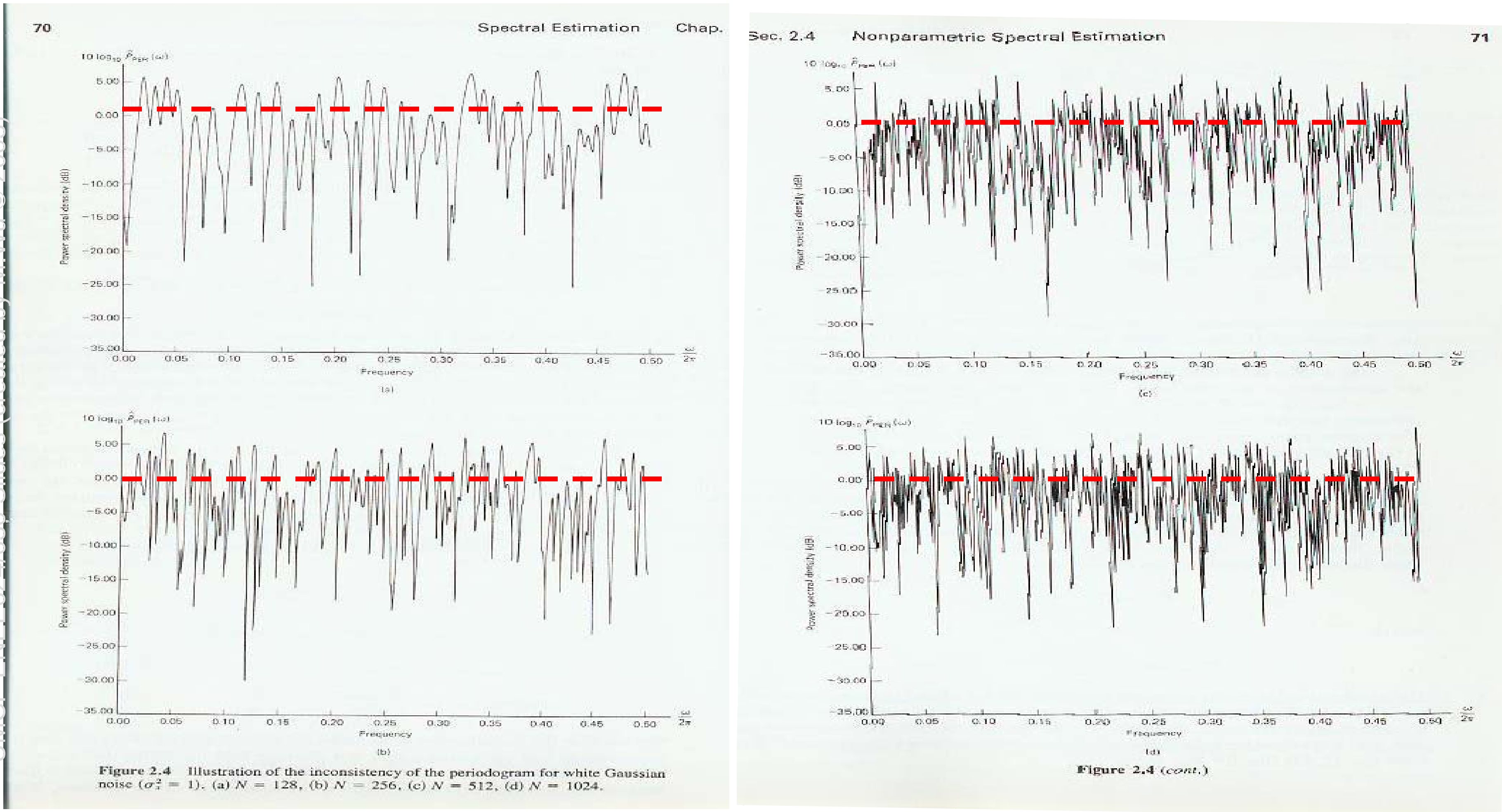
Periodogram: Filter Bank Perspective

- Can view the periodogram as an estimator of power spetrum that has a built-in filterbank
  - The filter bank ~ a set of bandpass filters
  - The estimated p.s.d. for each frequency f_0 is the power of one output sample of the bandpass filter centering at f_0

$$\hat{P}_{\mathrm{PER}}(f_0) = \left[N \cdot \left| \sum_{k=0}^{N-1} h[n-k]x[k] \right|^2 \right]_{n=0}$$

E.g. White Gaussian Process

[Lim/Oppenheim Fig.2.4]
Periodogram of zero-mean white Gaussian noise using N-point data record: N=128, 256, 512, 1024



- The random fluctuation (measured by variance) of the periodogram does not decrease with increasing N
  ➔ periodogram is not a consistent estimator

(3) How Good is Periodogram for Spectral Estimation?

$$\text{If } N \to \infty, \text{ will } \hat{P}_{\text{PER}} \to \text{p.s.d. } P(f)?$$

- Estimation: Tradeoff between bias and variance

$$E(\hat{\theta}) \stackrel{?}{=} \theta$$

$$E[\,|\hat{\theta} - E(\hat{\theta})|^2\,] = ?$$

- For white Gaussian process, we can show that at $f_k = k/N$

$$\Rightarrow \quad E[\hat{P}_{PER}(f_k)] = P(f_k), \quad k = 0, 1, \ldots N/2$$

$$Var[\hat{P}_{PER}(f_k)] = \begin{cases} P^2(f_k), & k = 1, \ldots \frac{N}{2} - 1 \\ 2P^2(f_k), & k = 0, \frac{N}{2} \end{cases} \quad \propto P^2(f_k)$$

Performance of Periodogram: Summary

- The periodogram for white Gaussian process is an unbiased estimator but not consistent
  - The variance does not decrease with increasing data length
  - Its standard deviation is as large as the mean (equal to the quantity to be estimated)
- Reasons for the poor estimation performance
  - Given N real data points, the # of unknown parameters $\{P(f_0), \ldots P(f_{N/2})\}$ we try to estimate is N/2, i.e. proportional to N
- Similar conclusions can be drawn for processes with arbitrary p.s.d. and arbitrary frequencies
  - Asymptotically unbiased (as N goes to infinity) but inconsistent

3.1.2 Averaged Periodogram

- As one solution to the variance problem of periodogram
  - Average K periodograms computed from K sets of data records

$$\hat{P}_{\text{AV PER}}(f) = \frac{1}{K} \sum_{m=0}^{K-1} \hat{P}^{(m)}_{\text{PER}}(f)$$

where $$\hat{P}^{(m)}_{\text{PER}}(f) = \frac{1}{L} \left| \sum_{n=0}^{L-1} x_m[n] e^{-j2\pi fn} \right|^2$$

And the K sets of data records are

$$\{x_0[0], \ldots, x_0[L-1]; \; x_1[n], \; 0 \le n \le L-1; \ldots$$
$$\{x_{K-1}[n-1], \; 0 \le n \le L-1\}$$

Performance of Averaged Periodogram

– If K sets of data records are uncorrelated with each other, we have: *($f_i = i/L$)*

$\hat{P}_{PER}^{(m)}(f)$ i.i.d. (m=0,1, … L-1) for white Gaussian process

$$\Rightarrow Var[\hat{P}_{AVPER}(f)] = \quad \propto \frac{1}{K} P^2(f_i)$$

Performance of Averaged Periodogram

– If K sets of data records are uncorrelated with each other, we have : $(f_i = i/L)$

$\hat{P}_{PER}^{(m)}(f)$ i.i.d. (m=0,1, … L-1) for white Gaussian process

$$\Rightarrow Var[\hat{P}_{AVPER}(f)] = \begin{cases} \frac{1}{K} P^2(f_i) & i = 1, 2, \cdots \frac{L}{2}-1 \\ \frac{2}{K} P^2(f_i) & i = 0, \frac{L}{2} \end{cases}$$

$$\propto \frac{1}{K} P^2(f_i)$$

i.e., $K \uparrow \rightarrow \text{Var} \downarrow$, and $\text{Var} \rightarrow 0$ for $K \rightarrow \infty$

i.e., consistent estimate

Practical Averaged Periodogram

- Usually we partition an available data sequence of length N
  - into K non-overlapping blocks, each block has length L (i.e. N=KL)

i.e. $$x_m[n] = x[n+mL], \qquad n = 0, 1, \ldots, L-1$$
$$m = 0, 1, \ldots, K-1$$

- Since the blocks are contiguous, the K sets of data records may not be completely uncorrelated
  - Thus the variance reduction factor is in general less than K

- Periodogram averaging is also known as the Bartlett's method

Averaged Periodogram for Fixed Data Size

- Given a data record of fixed size N, will the result be better if we segment the data into more and more subrecords?

We examine for a <u>real-valued</u> stationary process:

$$E\left[\hat{P}_{\text{AV PER}}(f)\right] = E\left[\frac{1}{K}\underbrace{\sum_{m=0}^{K-1}\hat{P}^{(m)}_{\text{PER}}(f)}_{\text{identical stat. mean for all } m}\right] = E\left[\hat{P}^{(0)}_{\text{PER}}(f)\right]$$

Note
$$\hat{P}^{(0)}_{\text{PER}}(f) = \sum_{l=-(L-1)}^{L-1}\hat{r}^{(0)}(l)e^{-j2\pi fl}$$

$\rightarrow$ an equivalent expression to definition in terms of x[n]

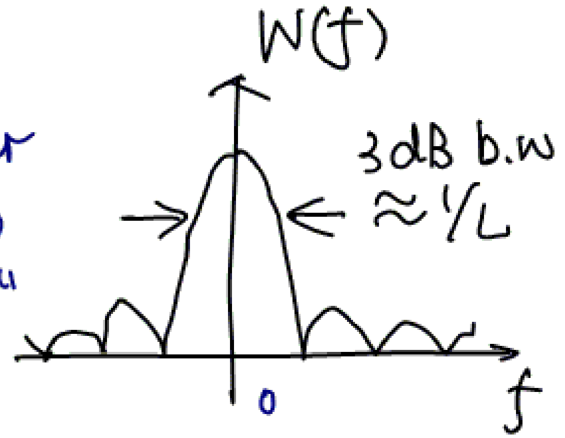
where
$$\hat{r}^{(0)}(l) = \frac{1}{L}\sum_{n=0}^{L-1-|l|}x[n]x\big[n+|l|\big]$$

Mean of Averaged Periodogram

$$W[k] = \begin{cases} 1 - |k|/L & \text{for } |k| \le L-1 \\ 0 & \text{o.w.} \end{cases}$$

"triangular (Barlett) window"

$$\Rightarrow W(f) = \frac{1}{L}\left(\frac{\sin \pi f L}{\sin \pi f}\right)^2$$



Mean of Averaged Periodogram

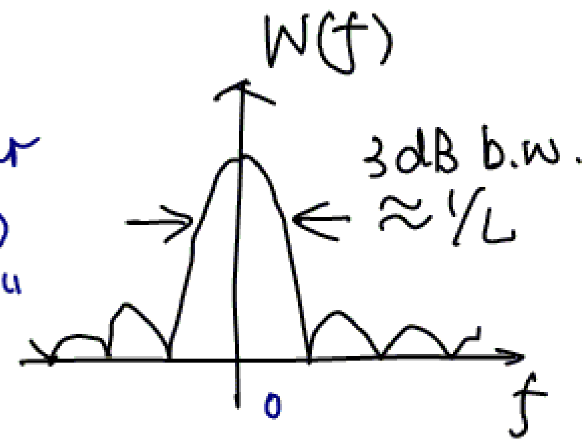
$$\Rightarrow E[\hat{r}^{(0)}(\ell)] = \underbrace{\left(1 - \frac{|\ell|}{L}\right)}_{\triangleq w[\ell]} r(\ell) \quad \text{for } |\ell| \le L-1$$

$$\therefore E[\hat{P}_{AVPER}(f)] = \sum_{\ell=-(L-1)}^{L-1} w[\ell]\, r(\ell)\, e^{-j2\pi f \ell}$$

$$w[k] = \begin{cases} 1 - |k|/L & \text{for } |k| \le L-1 \\ 0 & \text{o.w.} \end{cases}$$

"triangular (Barlett) window"

$$\Rightarrow W(f) = \frac{1}{L}\left(\frac{\sin \pi f L}{\sin \pi f}\right)^2$$



Mean of Averaged Periodogram (cont'd)

$$E[\hat{P}_{\text{AV PER}}(f)] = \text{DTFT}\left[\{w[k]r(k)\}\right]\Big|_f$$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} W(f-\eta)P(\eta)d\eta$$

$$\neq P(f)$$

multiplication in time

$\downarrow$

convolution in frequency

- Biased estimate (both averaged and regular periodogram)
  - The convolution with the window function w[k] lead to the mean of the averaged periodogram being smeared from the true p.s.d

Mean of Averaged Periodogram (cont'd)

$$E[\hat{P}_{\text{AV PER}}(f)] = \text{DTFT}\big[\{w[k]r(k)\}\big]\Big|_f$$

multiplication in time

↓

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} W(f-\eta)P(\eta)d\eta$$

convolution in frequency

$$\neq P(f)$$

- Biased estimate (both averaged and regular periodogram)
  - The convolution with the window function w[k] lead to the mean of the averaged periodogram being smeared from the true p.s.d
- Asymptotic unbiased as L $\rightarrow\infty$
  - To avoid the smearing, the window length L must be large enough so that the narrowest peak in P(f) can be resolved
- This gives a tradeoff between bias and variance

  Small K => better resolution (smaller smearing/bias) but larger variance

Non-parametric Spectrum Estimation: Recap

- Periodogram
  - Motivated by relation between p.s.d. and squared magnitude of DTFT of a finite-size data record
  - Variance: won't vanish as data length N goes infinity ~ "inconsistent"
  - Mean: asymptotically unbiased w.r.t. data length N in general
    - *equivalent to apply triangular window to autocorrelation function*
      (windowing in time gives smearing/smoothing in freq. domain)
    - *unbiased for white Gaussian (flat spectrum)*
- Averaged periodogram
  - Reduce variance by averaging K sets of data record of length L each
  - Small L increases smearing/smoothing in p.s.d. estimate thus higher bias ➔ *equiv. to triangular windowing*
- Windowed periodogram: generalize to other symmetric windows

Case Study on Non-parametric Methods

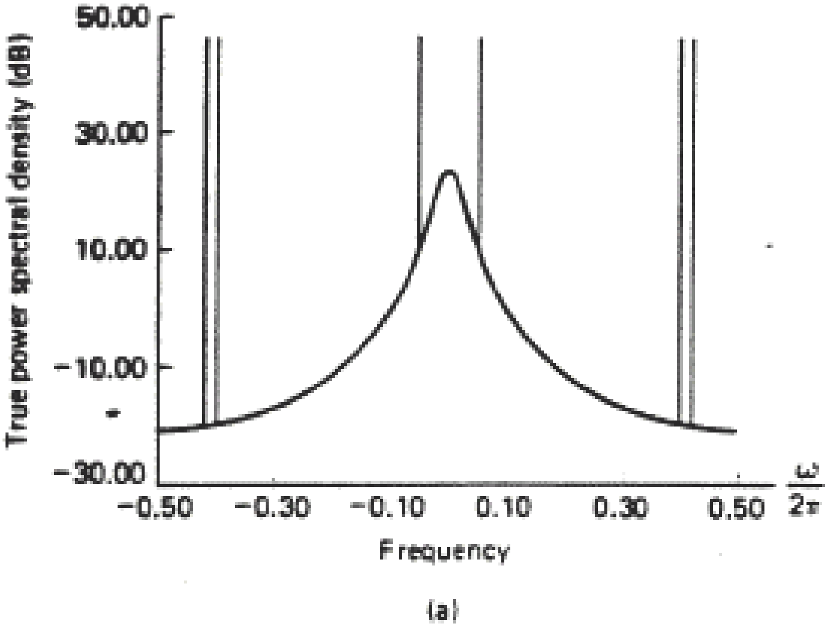
- Test case: a process consists of narrowband components (sinusoids) and a broadband component (AR)
  - $x[n] = 2\cos(\omega_1 n) + 2\cos(\omega_2 n) + 2\cos(\omega_3 n) + z[n]$

    where $z[n] = -a_1 z[n-1] + v[n]$, $a_1 = -0.85$, $\sigma^2 = 0.1$

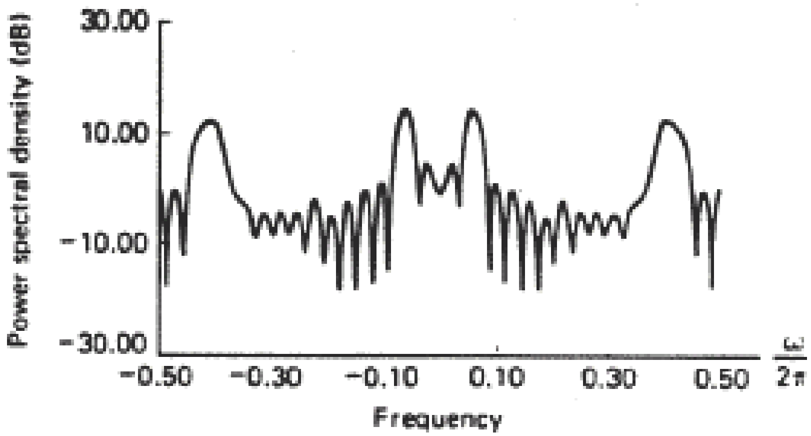
    $$\omega_1/2\pi = 0.05,\ \omega_2/2\pi = 0.40,\ \omega_3/2\pi = 0.42$$
  - N=32 data points are available

    ➔ periodogram resolution f = 1/32

- Examine typical characteristics of various non-parametric spectral estimators

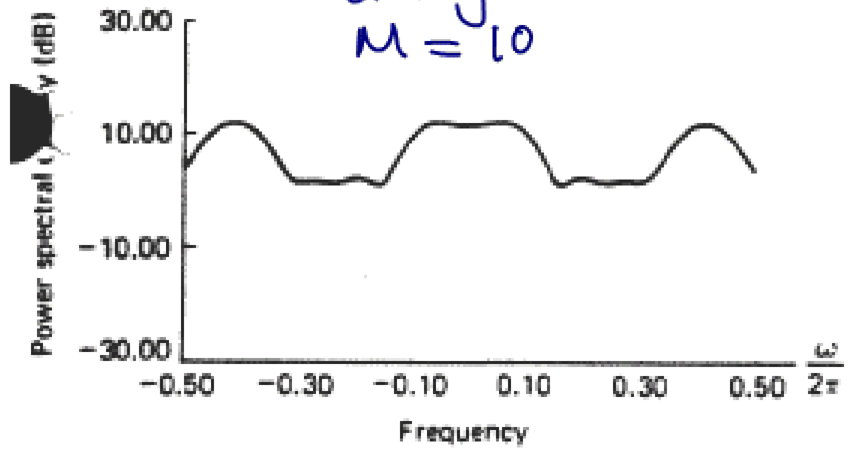


(Fig.2.17 from Lim/Oppenheim book)



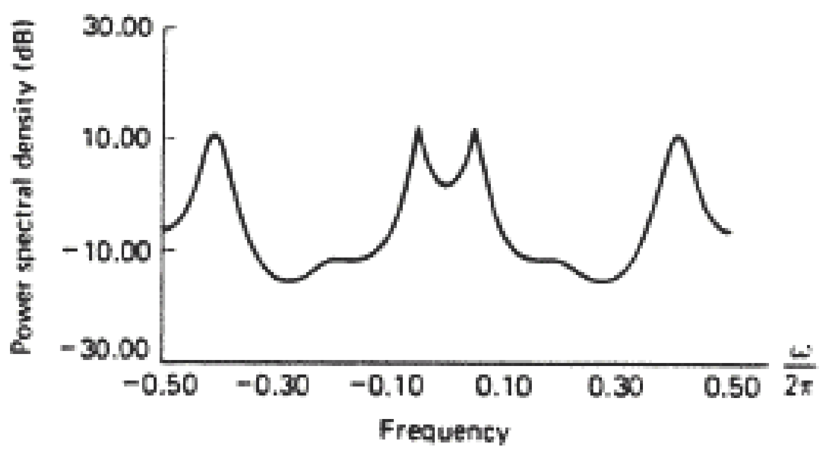
(b) Periodogram

triangle window
M = 10

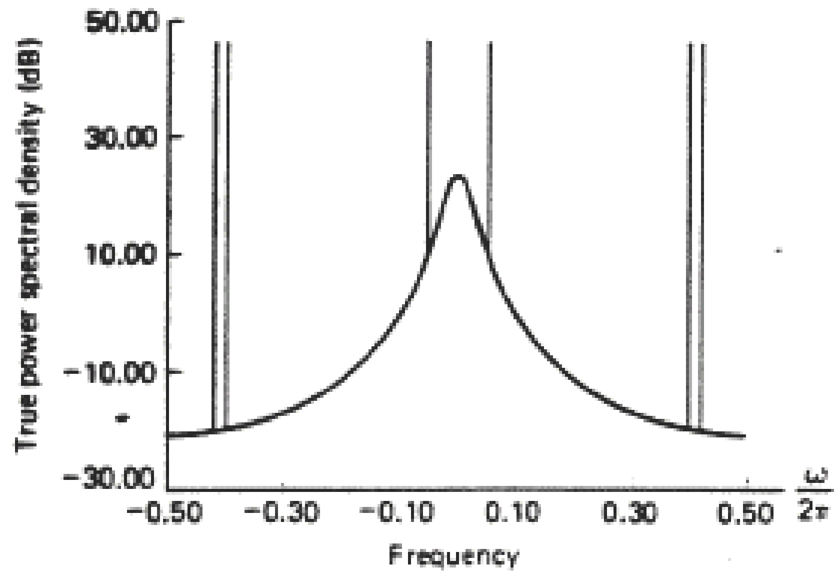


(c) Blackman–Tukey

f



(d) Minimum variance spectral estimator



true p.s.d.

(a)

3.1.3 Periodogram with Windowing

- Review and Motivation

The periodogram estimator can be given in terms of $\hat{r}(k)$

$$\hat{P}_{\text{PER}}(f) = \sum_{k=-(N-1)}^{N-1} \hat{r}(k) e^{-j2\pi fk}$$

where $$\hat{r}(k) = \frac{1}{N} \sum_{n=0}^{N-1-k} x^*[n] x[n+k]; \quad \hat{r}(-k) = \hat{r}^*(k) \quad \text{for } k \geq 0$$

  - The higher lags of r(k), the poorer estimates since the estimates involve fewer terms of lag products in the averaging operation

- Solution: weigh the higher lags less
  - Trade variance with bias

Windowing

- Use a window function to weigh the higher lags less

i.e. $$\hat{P}_{win}(f) = \sum_{k=-(N-1)}^{N-1} w[k]\,\hat{r}(k)\,e^{-j2\pi f k}$$

where $w[k]$ is a "lag window" with properties of:

① $0 \le w[k] \le w[0] = 1$

w(0)=1 preserves variance r(0)

② $w[-k] = w[k]$ symmetric

③ $w[k] = 0$ for $|k| > M$ where $M \le N-1$

④ $W(f)$ must be chosen to ensure $\hat{P}_{win}(f) \ge 0$

- Effect: periodogram smoothing
  – Windowing in time ⇔ Convolution/filtering the periodogram
  – Also known as the Blackman-Tukey method

Common Lag Windows

- Much of the art in non-parametric spectral estimation is in choosing an appropriate window (both in type and length)

TABLE 2.1 COMMON LAG WINDOWS

Name	Definition	Fourier Transform
Rectangular	$w(k) = \begin{cases} 1, & \lvert k\rvert \le M \\ 0, & \lvert k\rvert > M \end{cases}$	$W(\omega) = W_R(\omega) = \dfrac{\sin \frac{\omega}{2}(2M+1)}{\sin \omega/2}$
Bartlett	$w(k) = \begin{cases} 1 - \dfrac{\lvert k\rvert}{M}, & \lvert k\rvert \le M \\ 0, & \lvert k\rvert > M \end{cases}$	$W(\omega) = W_B(\omega) = \dfrac{1}{M}\left(\dfrac{\sin M\omega/2}{\sin \omega/2}\right)^2$
Hanning	$w(k) = \begin{cases} \dfrac{1}{2} + \dfrac{1}{2}\cos\dfrac{\pi k}{M}, & \lvert k\rvert \le M \\ 0, & \lvert k\rvert > M \end{cases}$	$W(\omega) = \dfrac{1}{4}W_R(\omega - \pi/M) + \dfrac{1}{2}W_R(\omega) + \dfrac{1}{4}W_R(\omega + \pi/M)$
Hamming	$w(k) = \begin{cases} 0.54 + 0.46\cos\dfrac{\pi k}{M}, & \lvert k\rvert \le M \\ 0, & \lvert k\rvert > M \end{cases}$	$W(\omega) = 0.23\, W_R(\omega - \pi/M) + 0.54\, W_R(\omega) + 0.23\, W_R(\omega + \pi/M)$
Parzen	$w(k) = \begin{cases} 2\left(1 - \dfrac{\lvert k\rvert}{M}\right)^3 - \left(1 - 2\dfrac{\lvert k\rvert}{M}\right)^3, & \lvert k\rvert \le M/2 \\ 2\left(1 - \dfrac{\lvert k\rvert}{M}\right)^3, & \dfrac{M}{2} < k \le M \\ 0, & \lvert k\rvert > M \end{cases}$	$W(\omega) = \dfrac{8}{M^3}\left(\dfrac{3}{2}\dfrac{\sin^4 M\omega/4}{\sin^4 \omega/2} - \dfrac{\sin^4 M\omega/4}{\sin^2 \omega/2}\right)$

Table 2.1 common lag window (from Lim-Oppenheim book)

Discussion: Estimate r(k) via Time Average

- Normalizing the sum of (N-k) pairs

by a factor of 1/N ? v.s. by a factor of 1/(N-k) ?

<u>Biased</u> (low variance) <u>Unbiased</u> (may not non-neg. definite)

$$\hat{r}_1(k) = \frac{1}{N}\sum_{n=0}^{N-1-k} x[n+k]\, x^*[n]; \quad \hat{r}_2(k) = \frac{1}{N-k}\sum_{n=0}^{N-1-k} x[n+k]\, x^*[n]$$

$$E(\hat{r}_1(k)) = \qquad\qquad E(\hat{r}_2(k)) =$$

- Hints on showing the non-negative definiteness: using $\hat{r}_1(k)$ to construct correlation matrix
- For $\hat{r}_2(k)$: HW#8

Discussion: Estimate r(k) via Time Average

- Normalizing the sum of (N-k) pairs

by a factor of 1/N ? v.s. by a factor of 1/(N-k) ?

Biased (low variance) | Unbiased (may not non-neg. definite)

$$\hat{r}_1(k) = \frac{1}{N}\sum_{n=0}^{N-1-k} x[n+k]\, x^*[n]; \quad \hat{r}_2(k) = \frac{1}{N-k}\sum_{n=0}^{N-1-k} x[n+k]\, x^*[n]$$

$$E(\hat{r}_1(k)) = \frac{N-k}{N} r(k) \qquad E(\hat{r}_2(k)) = r(k)$$

- Hints on showing the non-negative definiteness: using $\hat{r}_1(k)$ to construct correlation matrix

$$\hat{R}_N = \mathbf{X}^H \mathbf{X}, \text{ where}$$

$$\mathbf{X} = \frac{1}{\sqrt{N}}\begin{bmatrix} x(0) & 0 & 0 & & \\ x(1) & x(0) & 0 & & \\ \vdots & \vdots & x(0) & \ddots & x(0) \\ x(N-1) & \vdots & \vdots & & \vdots \\ 0 & x(N-1) & & & \\ \vdots & 0 & \vdots & & \vdots \\ & \vdots & & & \end{bmatrix}$$

- For $\hat{r}_2(k)$: HW#8

3.1.4 Minimum Variance Spectral Estimation (MVSE)

- Recall: filter bank perspective of periodogram
  - The periodogram can be viewed as estimating the p.s.d. by forming a bank of narrowband filters with sinc-like response
  - The high sidelobe can lead to "leakage" problem:
    - *large output power due to p.s.d outside the band of interest*

- MVSE designs filters to minimize the leakage from out-of-band spectral components
  - Thus the shape of filter is dependent on the frequency of interest and data adaptive

    (unlike the identical filter shape for periodogram)

  - MVSE is also referred to as the Capon spectral estimator

Main Steps of MVSE Method

1. Design a bank of bandpass filters $H_i(f)$ with center frequency f_i so that
   - Each filter rejects the maximum amount of out-of-band power
   - And passes the component at frequency f_i without distortion

2. Filter the input process { x[n] } with each filter in the filter bank and estimate the power of each output process

3. Set the power spectrum estimate at frequency f_i to be the power estimated above divided by the filter bandwidth

Formulation of MVSE

The MVSE designs a filter H(f) for each frequency of interest f_0

minimize the output power

(i.e., to pass the components at f_0 w/o distortion)

Formulation of MVSE

The MVSE designs a filter H(f) for each frequency of interest f_0

minimize the output power

$$\rho = \int_{-\frac{1}{2}}^{+\frac{1}{2}} |H(f)|^2 P(f) df$$

subject to $$H(f_0) = 1$$

(i.e., to pass the components at f_0 w/o distortion)

Deriving MVSE Solutions

Output Power From H(f) filter

From the filter bank perspective of periodogram:

$$H(f) = \sum_{n=-(N-1)}^{0} h[n] e^{-j2\pi fn}$$

Thus

$$\rho = \int_{-\frac{1}{2}}^{\frac{1}{2}} \sum_{k=-(N-1)}^{0} h[k] e^{-j2\pi fk} \sum_{l=-(N-1)}^{0} h^*[l] e^{j2\pi fl} P(f) df$$

Equiv. to filter r(k) with { h(k) ⊗ h*(-k) } and evaluate at output time k=0

Output Power From H(f) filter

From the filter bank perspective of periodogram:

$$H(f) = \sum_{n=-(N-1)}^{0} h[n] e^{-j2\pi fn}$$

Thus

$$\rho = \int_{-\frac{1}{2}}^{\frac{1}{2}} \sum_{k=-(N-1)}^{0} h[k] e^{-j2\pi fk} \sum_{l=-(N-1)}^{0} h^*[l] e^{j2\pi fl} P(f) df$$

$$= \sum_{k=-(N-1)}^{0} \sum_{l=-(N-1)}^{0} h[k] h^*[l] \int_{-\frac{1}{2}}^{\frac{1}{2}} P(f) e^{j2\pi f(l-k)} df$$

$$= \sum_{k=-(N-1)}^{0} \sum_{l=-(N-1)}^{0} h[k] h^*[l] r(l-k)$$

Equiv. to filter r(k) with { h(k) ⊗ h*(-k) } and evaluate at output time k=0

Matrix-Vector Form of MVSE Formulation

Define

$$\underline{h}^* \triangleq \begin{bmatrix} h[0] \\ h[-1] \\ \vdots \\ h[-(N-1)] \end{bmatrix} \Rightarrow \rho = \underline{h}^H R^T \underline{h}$$

$$[h(0),\ h(-1) \ldots h(1-N)] \begin{bmatrix} r(0) & r(-1) & \cdots \\ r(1) & r(0) & \\ \vdots & & \ddots \end{bmatrix} \begin{bmatrix} h^*(0) \\ \vdots \\ \vdots \end{bmatrix}$$

$$\underline{e} = \begin{bmatrix} 1 \\ e^{j2\pi f_0} \\ \vdots \\ e^{j2\pi(N-1)f_0} \end{bmatrix}$$

➔ The constraint can be written in vector form as $\underbrace{\underline{h}^H \underline{e}}_{H(f_0)} = 1$

Matrix-Vector Form of MVSE Formulation

Define

$$\underline{h}^* \triangleq \begin{bmatrix} h[0] \\ h[-1] \\ \vdots \\ h[-(N-1)] \end{bmatrix} \Rightarrow P = \underline{h}^H R^T \underline{h}$$

$$[h(0), h(-1) \ldots h(1-N)] \begin{bmatrix} r(0) & r(-1) & \cdots \\ r(1) & r(0) & \\ \vdots & & \ddots \end{bmatrix} \begin{bmatrix} h^*(0) \\ \vdots \\ \vdots \end{bmatrix}$$

$$\underline{e} = \begin{bmatrix} 1 \\ e^{j2\pi f_0} \\ \vdots \\ e^{j2\pi(N-1)f_0} \end{bmatrix}$$

➔ The constraint can be written in vector form as $\underbrace{\underline{h}^H \underline{e}}_{H(f_0)} = 1$

Thus the problem becomes

$$\min_{\underline{h}} \underline{h}^H R^T \underline{h} \qquad \text{subject to} \qquad \underline{h}^H \underline{e} = 1$$

Solving MVSE

$$J \overset{def}{=} \underline{h}^H R^T \underline{h} + \mathrm{Re}\left[2\lambda(1-\underline{h}^H \underline{e})\right]$$

- Use Lagrange multiplier approach for solving the constrained optimization problem
  - Define **real-valued** objective function s.t. the stationary condition can be derived in a simple and elegant way based on the theorem for complex derivative/gradient operators

$$\min_{\underline{h},\lambda} J = \underline{h}^H R^T \underline{h} + \lambda(1-\underline{h}^H \underline{e}) + \left[\lambda(1-\underline{h}^H \underline{e})\right]^*$$

$$= \underline{h}^H R^T \underline{h} + \lambda(1-\underline{h}^H \underline{e}) + \lambda^*(1-\underline{e}^H \underline{h})$$

either $\nabla_{\underline{h}^*} J = 0 \Rightarrow R^T \underline{h} - \lambda \underline{e} = 0$

or $\nabla_{\underline{h}} J = 0 \Rightarrow \left(\underline{h}^H R^T\right)^T - \lambda^* \underline{e}^* = 0$

$$\Rightarrow \left(R^T\right)^H \underline{h} - \lambda \underline{e} = 0 \Rightarrow R^T \underline{h} - \lambda \underline{e} = 0$$

$$\Rightarrow \quad \underline{h} = \lambda\left(R^T\right)^{-1} \underline{e}$$

and $\underline{h}^H \underline{e} = 1$

Solution to MVSE

$$\min_{\underline{h},\lambda} J = \underline{h}^H R^T \underline{h} + \lambda(1-\underline{h}^H \underline{e}) + \left[\lambda(1-\underline{h}^H \underline{e})\right]^*$$

$$\begin{cases} \nabla_{\lambda^*} \text{ or } \nabla_{\lambda} J = 0 \Rightarrow \underline{h}^{\mathrm{H}} \underline{e} = 1 \quad (*) \\ \nabla_{\underline{h}^*} \text{ or } \nabla_{\underline{h}} J = 0 \Rightarrow R^T \underline{h} - \lambda \underline{e} = 0 \Rightarrow \underline{h} = \lambda (R^T)^{-1} \underline{e} \quad (**) \end{cases}$$

Bring $(**)$ into $(*)$:

$$\lambda = \frac{1}{\underline{e}^H \left(R^T\right)^{-1} \underline{e}}$$

Filter's output power:

$$\rho = \underline{h}^H R^T \underline{h} = \underline{h}^H R^T \left(R^T\right)^{-1} \underline{e}\, \lambda$$

$$= \lambda$$

The optimal filter and its output power:

$$\underline{h}_{MV} = \frac{\left(R^T\right)^{-1}}{\underline{e}^H \left(R^T\right)^{-1} \underline{e}}\ \underline{e}$$

$$\rho = \frac{1}{\underline{e}^H \left(R^T\right)^{-1} \underline{e}}$$

MVSE: Summary

If choosing the bandpass filters to be FIR of length q, its 3dB-b.w. is approximately 1/q

Thus the MVSE is

$$\hat{P}_{\mathrm{MV}}(f) = \frac{q}{\underline{e}^H \left(\hat{R}^T\right)^{-1} \underline{e}}$$

(i.e. normalize by filter b.w.)

$\hat{R}$ is $q \times q$ correlation matrix

$$\underline{e} = \begin{bmatrix} 1 \\ \exp(j2\pi f) \\ \vdots \\ \exp(j2\pi f(q-1)) \end{bmatrix}$$

- MVSE is a data adaptive estimator and provides improved resolution and reduced variance over periodogram
  - Also referred to as "High-Resolution Spectral Estimator"
  - Doesn't assume a particular underlying model for the data

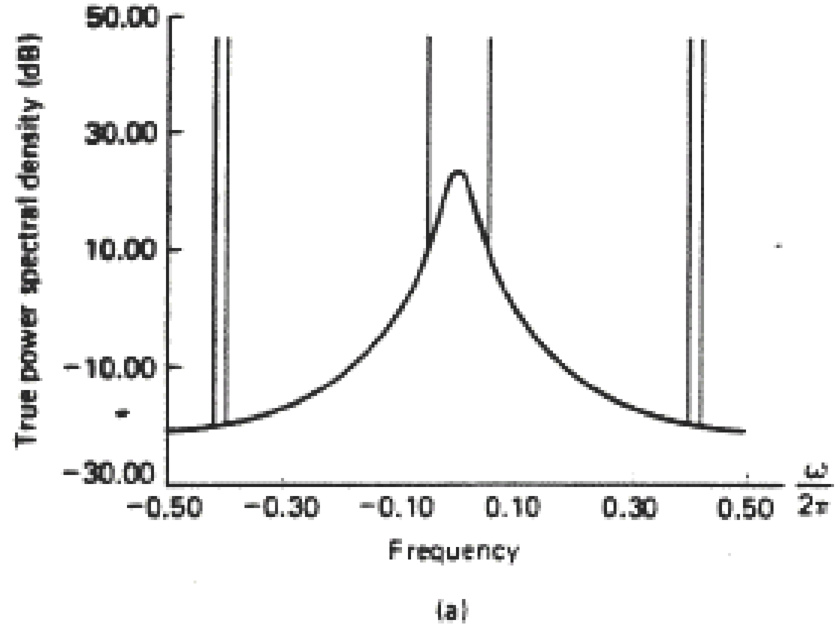
MVSE vs. Periodogram

- MVSE is a data adaptive estimator and provides improved resolution and reduced variance over periodogram

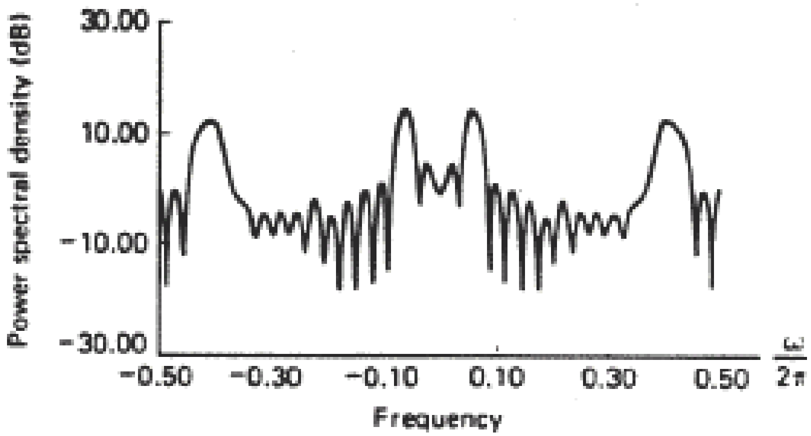
	Periodogram	**MVSE**
Equivalent Bandpass Filter $\underline{h}$	$\underline{e}$ Filter is "universal" data-independent	$\dfrac{\left(R^T\right)^{-1}}{\underline{e}^H\left(R^T\right)^{-1}\underline{e}}\,\underline{e}$ Filter adapts to observation data via *R*
Equivalent spectrum estimate $\hat{P}(f)$	$q\cdot\underline{e}^H\hat{R}^T\underline{e}$	$\dfrac{q}{\underline{e}^H\left(\hat{R}^T\right)^{-1}\underline{e}}$

Recall: Case Study on Non-parametric Methods

- Test case: a process consists of narrowband components (sinusoids) and a broadband component (AR)
  – $x[n] = 2\cos(\omega_1 n) + 2\cos(\omega_2 n) + 2\cos(\omega_3 n) + z[n]$
    where $z[n] = -a_1 z[n-1] + v[n]$, $a_1 = -0.85$, $\sigma^2 = 0.1$
    $\omega_1/2\pi = 0.05$, $\omega_2/2\pi = 0.40$, $\omega_3/2\pi = 0.42$
  – N=32 data points are available
    ➔ periodogram resolution f = 1/32
- Examine typical characteristics of various non-parametric spectral estimators

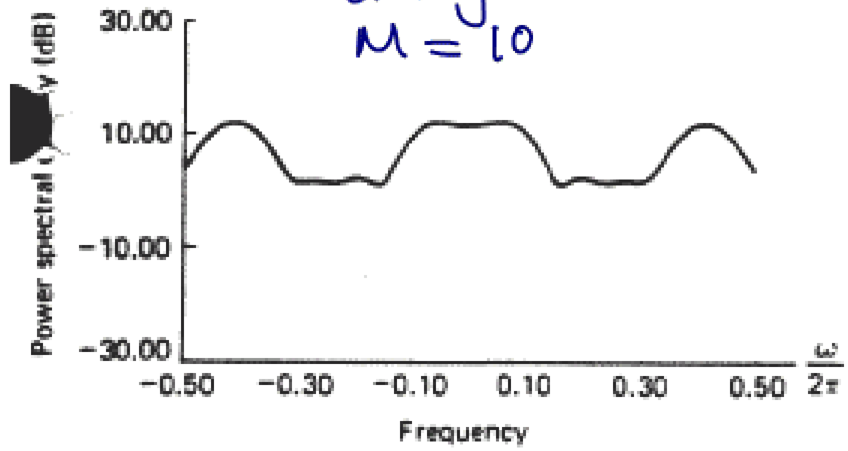


(Fig.2.17 from Lim/Oppenheim book)

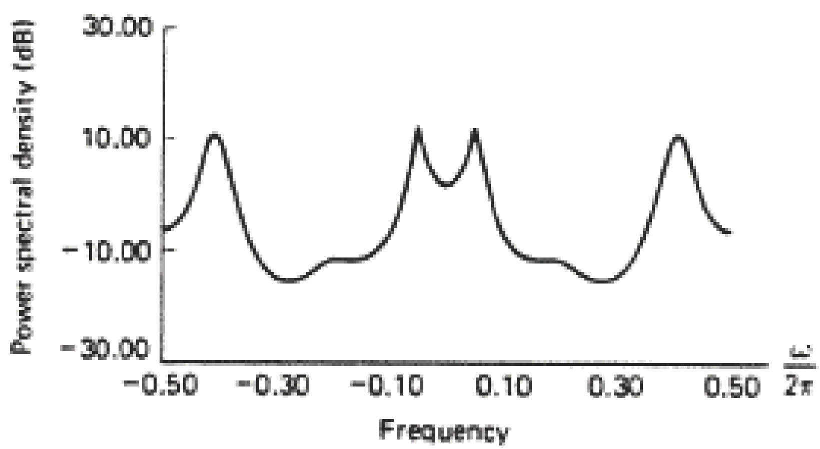


(b) Periodogram

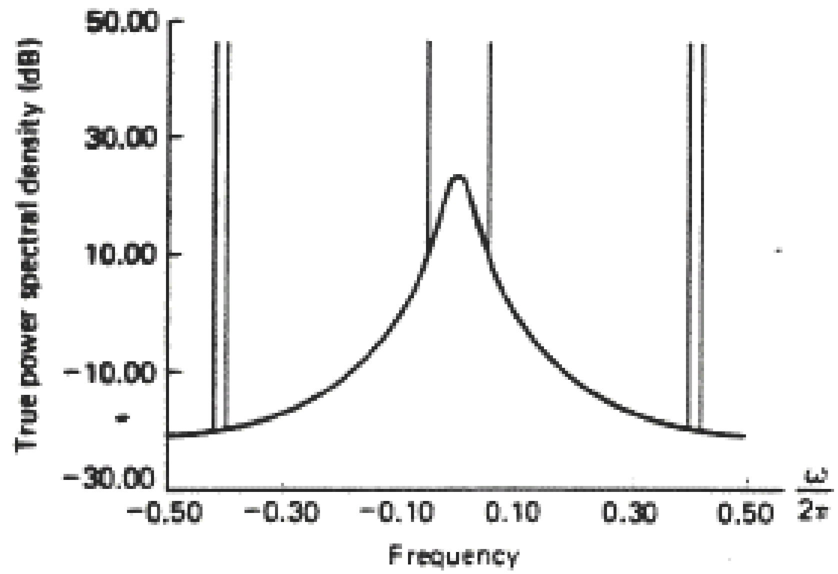
triangle window
$M = 10$



(c) Blackman-Tukey



(d) Minimum variance spectral estimator



true p.s.d.

(a)

Ref. on Derivative and Gradient Operators for Complex-Variable Functions

Ref: D.H. Brandwood, “A complex gradient operator and its application in adaptive array theory,” in IEE Proc., vol. 130, Parts F and H, no.1, Feb. 1983.

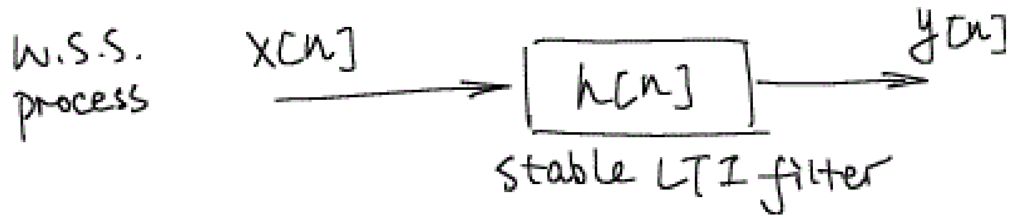
(downloadable from IEEEXplorer)

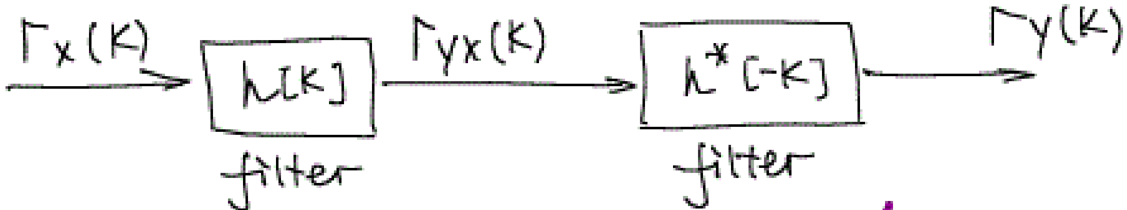
– Solving constrained optimization with real-valued objective function of complex variables, subject to constraint function of complex variables

- *As seen in minimum variance spectral estimation and other array/statistical signal processing context.*

Reference

Recall: Filtering a Random Process





$$\Gamma_h[k] = h[k] * h^*[-k] = \sum_{l=-\infty}^{+\infty} h[l]\, h^*[k+l]$$

In terms of ZT:

$h^*[-k]$

$$P_y(z) = P_x(z)\, H(z)\, H^*(1/z^*)$$

$$\underset{z=e^{j\omega}}{\Rightarrow} P_y(\omega) = P_x(\omega)\, H(\omega)\, H^*(\omega) = P_x(\omega)\, |H(\omega)|^2 .$$

Chi-Squared Distribution

If $x[n] \sim$ iid $N(0,1)$ for $n = 0, 1, \ldots N-1$, and

$$y = \sum_{n=0}^{N-1} x^2[n],$$

then y follows chi-squared distribution of degree N, i.e. $y \sim \chi_N^2$

and $E[y] = N$, $Var(y) = 2N$

Chi-Squared Distribution (cont'd)

p.d.f. of $y \sim \chi_N^2$:

$$p(y) = \begin{cases} \dfrac{1}{2^{N/2}\,\Gamma(N/2)}\, y^{\frac{N}{2}-1} e^{-\frac{y}{2}} & \text{if } y \geq 0 \\ 0 & \text{if } y < 0 \end{cases}$$

where $\Gamma(\cdot)$ is the gamma integral

$$\Gamma(x+1) = \int_0^\infty y^x e^{-y}\, dy \quad \text{for } x > -1 .$$

Note if x is an integer, $\Gamma(n+1) = n\Gamma(n) = n!$

Periodogram of White Gaussian Process

For $f_k = k/N$, it can be shown that

$$
\begin{cases}
\dfrac{2\hat{P}_{PER}(f_k)}{P(f_k)} \sim \chi_2^2 & \text{for } k = 1, 2, \cdots \frac{N}{2} - 1; \\[2ex]
\dfrac{\hat{P}_{PER}(f_k)}{P(f_k)} \sim \chi_1^2 & \text{for } k = 0, \frac{N}{2}
\end{cases}
$$

$$
\Rightarrow \quad E[\hat{P}_{PER}(f_k)] = P(f_k), \quad k = 0, 1, \cdots N/2
$$

$$
Var[\hat{P}_{PER}(f_k)] = \begin{cases} P^2(f_k), & k = 1, \cdots \frac{N}{2} - 1 \\ 2P^2(f_k), & k = 0, \frac{N}{2} \end{cases}
$$

See proof in Appendix 2.1 in Lim-Oppenheim Book:
- Basic idea is to examine the distribution of real and imaginary part of the DFT, and take the magnitude

Preview & Warm-up

Model or Not?

- Implicit assumption by classical methods
  - Classical methods use Fourier transform on either windowed data or windowed autocorrelation function (ACF)
  - Implicitly assume the unobserved data or ACF outside the window are zero => not true in reality
  - Consequence of windowing: smeared spectral estimate (leading to low resolution)
- If prior knowledge about the process is available
  - Can use prior knowledge and select a good model to approximate the process
  - Usually need to estimate fewer model parameters (than non-parametric approaches) using the limited data points we have
  - The model may allow to better describe the process outside the window (instead of assuming zeros)

General Procedure of Parametric Methods

- Select a model (based on prior knowledge)
- Estimate the parameters of the assumed model
- Obtain the spectral estimate implied by the model (with the estimated parameters)

Spectral Estimation using AR, MA, ARMA Models

- Physical insight: the process is generated/approximated by filtering white noise with an LTI filter of rational transfer func H(z)
- Use observed data to estimate a few lags of r(k)
  - Larger lags of r(k) can be implicitly extrapolated by the model
- Relation between r(k) and filter parameters $\{a_k\}$ and $\{b_k\}$
  - PARAMETER EQUATIONS from Section 2.1.2(6) ← review this
  - Solve the parameter equations to obtain filter parameters
  - Use the p.s.d. implied by the model as our spectral estimate
- Deal with nonlinear parameter equations
  - Try to "convert" or relate them to AR models that has linear equations

Review: Parameter Equations

Yule-Walker equations (for AR process)

$$r_x[k] = \begin{cases} -\sum_{l=1}^{p} a[l]\, r_x[-l] + \sigma^2 & \text{for } k=0 \\ -\sum_{l=1}^{p} a[l]\, r_x[k-l] & \text{for } k \geq 1 \end{cases}$$

$$\begin{bmatrix} r_x(0) & r_x(-1) & \cdots & r_x(-p+1) \\ r_x(1) & r_x(0) & \cdots & r_x(-p+2) \\ \vdots & & \ddots & \vdots \\ r(p-1) & \cdots & & r(0) \end{bmatrix} \begin{bmatrix} a[1] \\ a[2] \\ \vdots \\ a[p] \end{bmatrix} = - \begin{bmatrix} r_x(1) \\ r_x(2) \\ \vdots \\ r_x(p) \end{bmatrix}$$

ARMA model

$$r_x[k] = \begin{cases} -\sum_{l=1}^{p} a[l]\, r_x[k-l] + \sigma^2 \sum_{l=0}^{q-k} h^*[l]\, b[l+k] & k=0,1,\cdots q \\ -\sum_{l=1}^{p} a[l]\, r_x[k-l] \;, & k \geq q+1 . \end{cases}$$

MA model

$$r_x[k] = \begin{cases} \sigma^2 \sum_{l=0}^{q-k} b^*[l]\, b[l+k] & \text{for } k=0,1,\cdots q \\ 0 & \text{for } k \geq q+1 \end{cases}$$

Spectrum Estimation with AR Modeling

- Use Levinson-Durbin recursion and solve for

$$\begin{bmatrix} \hat{r}(0) & \hat{r}(1) & \cdots & \hat{r}(L-1) \\ \hat{r}(1) & \hat{r}(0) & & \hat{r}(L-2) \\ \vdots & & \ddots & \vdots \\ \hat{r}(L-1) & \cdots & \cdots & \hat{r}(0) \end{bmatrix} \begin{bmatrix} \hat{a}_1 \\ \hat{a}_2 \\ \vdots \\ \hat{a}_L \end{bmatrix} = - \begin{bmatrix} \hat{r}(1) \\ \vdots \\ \hat{r}(L) \end{bmatrix}$$

where $$\hat{r}(k) = \frac{1}{N} \sum_{n=0}^{N-1-k} x[n]\, x[n+k]$$

  - Approximate the observed data sequence {x[0], …, x[N]} with an AR model (consider real-valued process here for simplicity)
  - Use biased ACF estimate here to ensure nonnegative definiteness and smaller variance than unbiased estimate (dividing by N-k)

MA Spectral Estimation

An MA(q) model

$$x[n] = \sum_{k=0}^{q} b_k v[n-k] \quad \Rightarrow \quad B(z) = \sum_{k=0}^{q} b_k z^{-k}$$

can be used to define an MA spectral estimator

$$\hat{P}_{\mathrm{MA}}(f) = \sigma^2 \left| 1 + \sum_{k=1}^{q} b_k e^{-j2\pi fk} \right|^2$$

Recall:

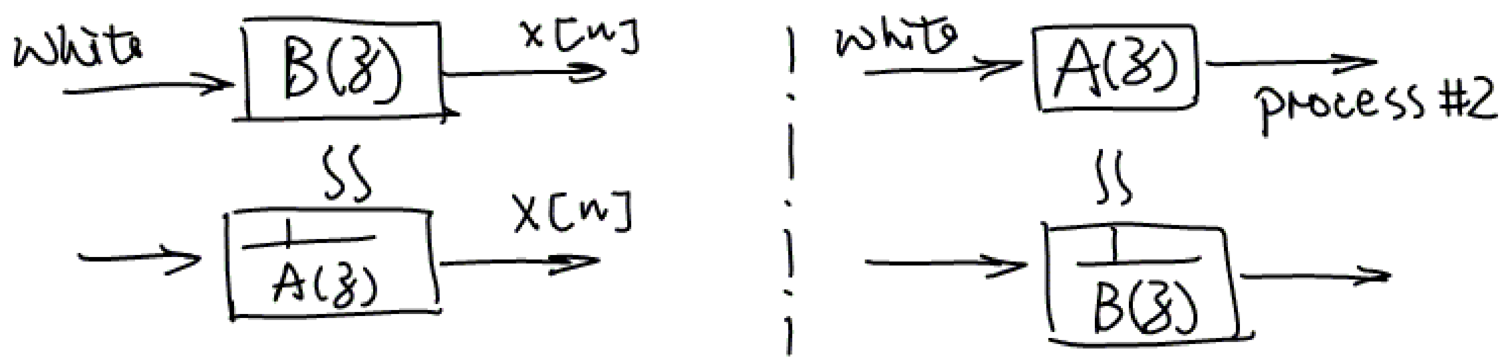
(1) The problem of solving for b_k given {r(k)} is to solve a set of nonlinear equations;

(2) An MA process can be approximated by an AR process of sufficiently high order.

Basic Idea to Avoid Solving Nonlinear Equations

Consider two processes:

- Process#1: we observed N samples, and need to perform spectral estimate
  - We first model it as a high-order AR process, generated by 1/A(z) filter



- Process#2: an MA-process generated by A(z) filter
  - Since we know A(z), we can know process#2's autocorrelation function;
  - We model process#2 as an AR(q) process => the filter would be 1/B(z)

Complex Exponentials in Additive Noise

$$x[n] = \sum_{i=1}^{p} A_i e^{j\phi_i} e^{j2\pi f_i n} + w[n]$$

$n = 0, 1, \ldots N-1$ (Observe N samples)

$w[n]$ — white noise, zero mean, variance σ_w^2

A_i, f_i — real, constant, unknown

$\Rightarrow$ to be estimated

ϕ_i — uniform distribution over $[0, 2\pi)$; uncorrelated with $w[n]$ and between different i.

Correlation Matrix for the Process

– Determine autocorrelation function

– Rs = ? Rw = ? Rx = ?

– Rank of correlation matrices?

Correlation Matrix for the Process

$$r_x(k) = E[x[n]x^*[n-k]] = \sum_{i=1}^{p} \underbrace{A_i^2}_{\triangleq P_i} e^{j2\pi f_i k} + \sigma_w^2 \delta(k)$$

An $M \times M$ correlation matrix for $\{x[n]\}$ $(M > p)$:

$$R_x = R_s + R_w$$

$$R_w = \sigma_w^2 I \longrightarrow \text{full rank}$$

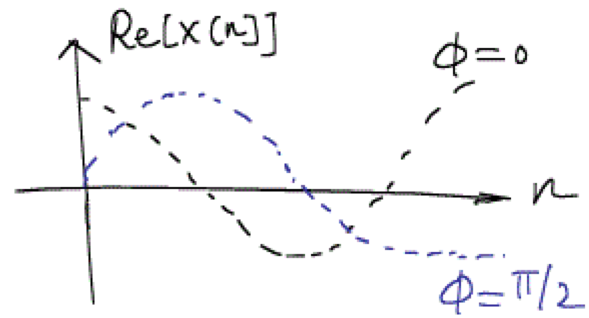
$$R_s = \sum_{i=1}^{p} P_i \, \underline{e}_i \, \underline{e}_i^H$$

where $\underline{e}_i = [1, e^{-j2\pi f_i}, e^{-j4\pi f_i}, \ldots e^{-j2\pi f_i(M-1)}]^T$

Correlation Matrix for the Process

$$x[n] = A \exp[j(2\pi f_0 n + \phi)]$$

$$E[x[n]] = 0 \quad \forall n$$



$$E[x[n]\, x[n-k]]$$

$$= E[A \exp[j(2\pi f_0 n + \phi)] \cdot A \exp[-j(2\pi f_0 n - 2\pi f_0 k + \phi)]]$$

$$= A^2 \cdot \exp[j(2\pi f_0 k)]$$

$\therefore$ $x[n]$ is zero-mean w.s.s. with $r_x(k) = A^2 \exp(j 2\pi f_0 k)$.

$y[n] = x[n] + w[n]$ <u>white noise</u>: $E[w[n]\, w^*[n-k]] = \begin{cases} \sigma^2 & k = 0 \\ 0 & \text{o.w.} \end{cases}$

$$r_y(k) = E[y[n]\, y^*[n-k]] = E[(x[n]+w[n])(x^*[n-k]+w^*[n-k])]$$

$$= r_x[k] + r_w[k] \quad (\because E[x[\cdot]\, w[\cdot]] = 0 \text{ uncorrelated})$$

$$= A^2 \exp[j 2\pi f_0 k] + \sigma^2 \delta[k]$$

E[x( ) w( )] = E[x( )] E[w( )] = 0
this crosscorr term vanish
because of uncorrelated *and*
zero mean for either x( ) or w( ).