ENEE630 Part-1 Supplement

Tree-based Filter Banks and Multiresolution Analysis

ECE Department

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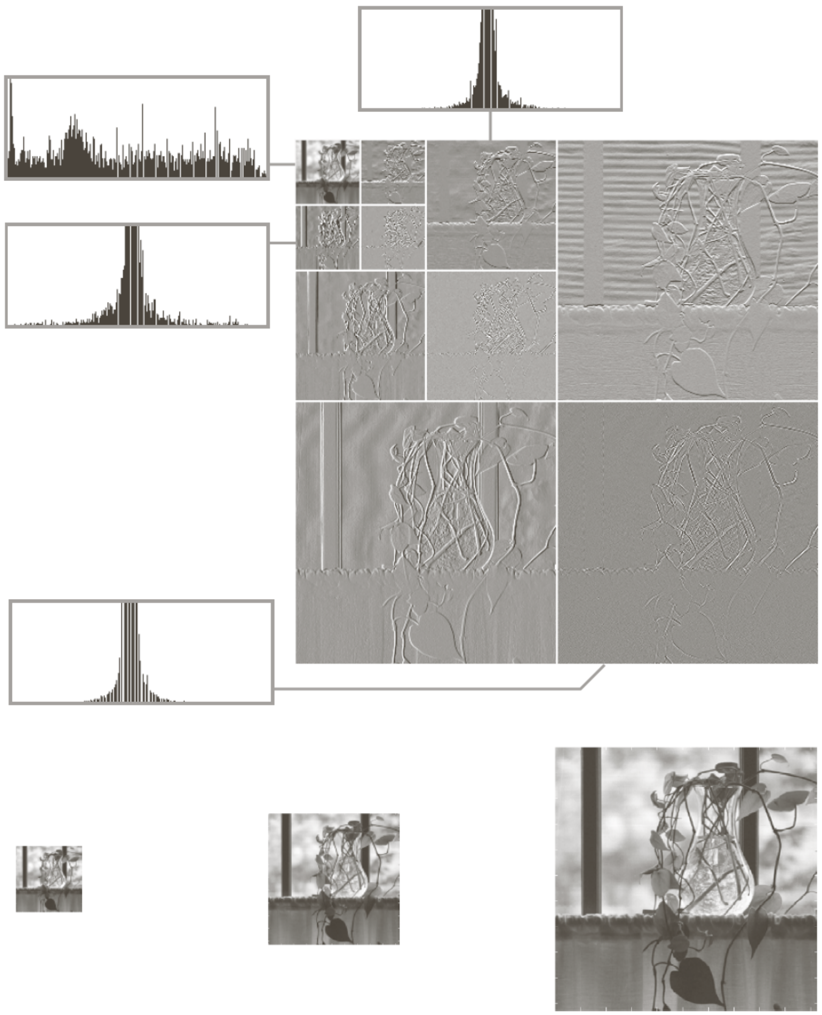
Dynamic Range of Original and Subband Signals



a
b c d

FIGURE 7.10 (a) A discrete wavelet transform using Haar $\mathbf{H}_2$ basis functions. Its local histogram variations are also shown. (b)–(d) Several different approximations (64×64, 128×128, and 256×256) that can be obtained from (a).

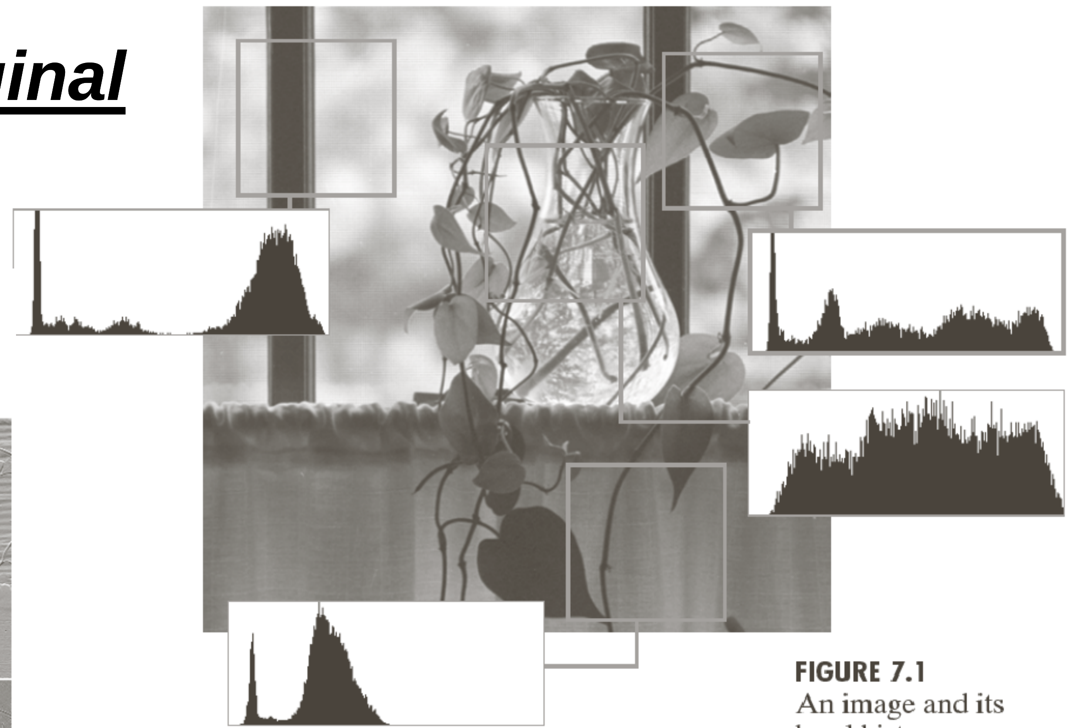


FIGURE 7.1 An image and its local histogram variations.

– Can assign more bits to represent coarse info

– Allocate remaining bits, if available, to finer details (via proper quantization)

Figures from Gonzalez/ Woods DIP 3/e book website.

Brief Note on Subband and Wavelet Coding

- The octave ("dyadic") frequency partition can reflect the logarithmatic characteristics in human perception
- Wavelet coding and subband coding have many similarities (e.g. from filter bank perspectives)
  - Traditionally subband coding uses filters that have little overlap to isolate different bands
  - Wavelet transform imposes smoothness conditions on the filters that usually represent a set of basis generated by shifting and scaling ("dilation") of a "mother wavelet" function
  - Wavelet can be motivated from overcoming the poor time-domain localization of short-time FT

Explore more in Proj#1. See PPV Book Chapter 11

Review and Examples of Basis

- Standard basis vectors

$$\begin{bmatrix}6\\3\\1\end{bmatrix} = 6\cdot\begin{bmatrix}1\\0\\0\end{bmatrix} + 3\cdot\begin{bmatrix}0\\1\\0\end{bmatrix} + 1\cdot\begin{bmatrix}0\\0\\1\end{bmatrix}$$

- Standard basis images

$$\begin{bmatrix}2 & 2\\3 & 0\end{bmatrix} = 2\cdot\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix} + 2\cdot\begin{bmatrix}0 & 1\\0 & 0\end{bmatrix} + 3\cdot\begin{bmatrix}0 & 0\\1 & 0\end{bmatrix} + 0\cdot\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}$$

- Example: representing a vector with different basis

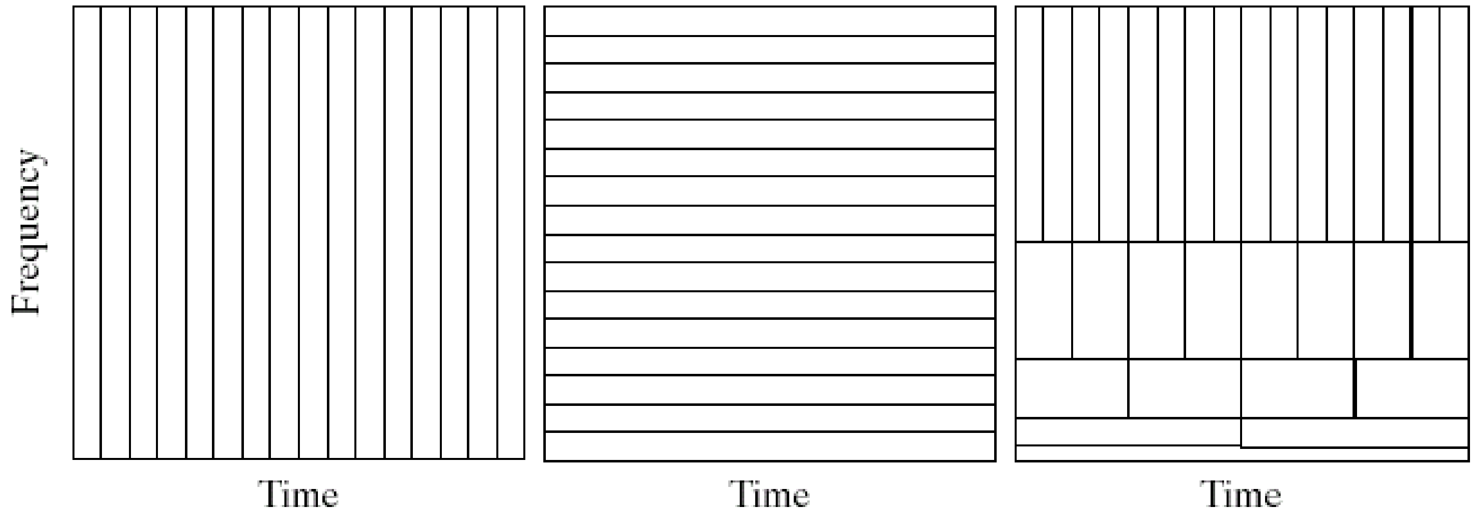
$$\begin{bmatrix}3\\5\end{bmatrix} = 3\cdot\begin{bmatrix}1\\0\end{bmatrix} + 5\cdot\begin{bmatrix}0\\1\end{bmatrix} = 4\cdot\begin{bmatrix}1\\1\end{bmatrix} + 1\cdot\begin{bmatrix}-1\\1\end{bmatrix} = 4\sqrt{2}\begin{bmatrix}\sqrt{2}/2\\\sqrt{2}/2\end{bmatrix} + \sqrt{2}\begin{bmatrix}-\sqrt{2}/2\\\sqrt{2}/2\end{bmatrix}$$

Time-Freq (or Space-Freq) Interpretations

– Inverse transf. represents a signal as a linear combination of basis vectors

– Forward transf. determines combination coeff. by projecting signal onto basis

E.g. Standard Basis (for data samples); Fourier Basis; Wavelet Basis



a b c

FIGURE 7.21 Time-frequency tilings for (a) sampled data, (b) FFT, and (c) FWT basis functions.

Figures from Gonzalez/ Woods DIP 2/e book website.

Recall: Matrix/Vector Form of DFT

$$\begin{cases} Z(k) = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} z(n) \cdot W_N^{nk} \\ z(n) = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} Z(k) \cdot W_N^{-nk} \end{cases}$$

- { z(n) } ⇔ { Z(k) }

  n, k = 0, 1, …, N-1, $W_N = exp\{ -j2\pi / N \}$

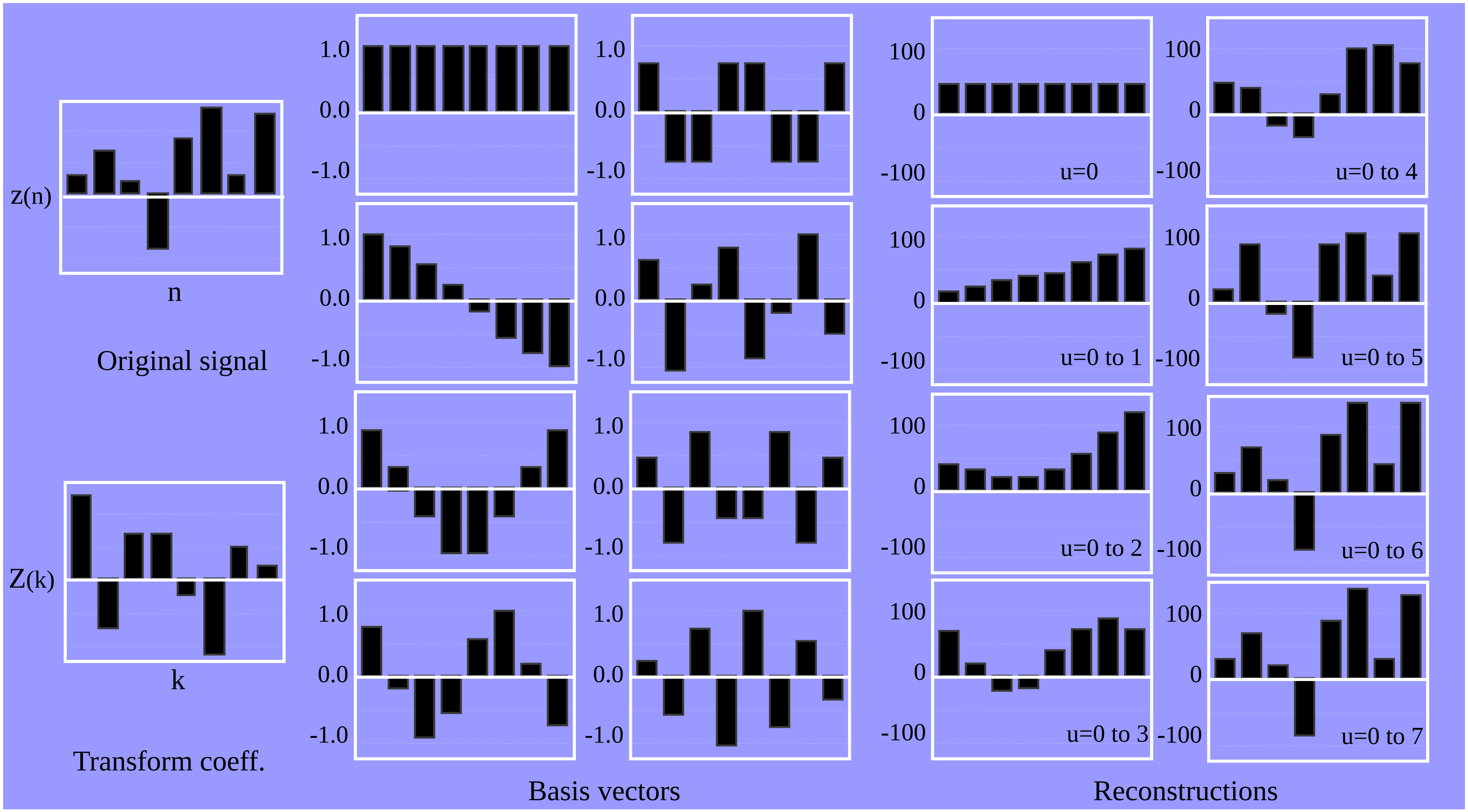
  ~ complex conjugate of primitive N[th] root of unity

$$\begin{bmatrix} Z(0) \\ Z(1) \\ Z(2) \\ \vdots \\ Z(N-1) \end{bmatrix} = \frac{1}{\sqrt{N}} \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & e^{-j2\pi/N} & e^{-j2\pi\cdot 2/N} & \cdots & e^{-j2\pi\cdot(N-1)/N} \\ 1 & e^{-j2\pi\cdot 2/N} & e^{-j2\pi\cdot 4/N} & \cdots & e^{-j2\pi\cdot 2(N-1)/N} \\ \vdots & \vdots & & & \vdots \\ 1 & e^{-j2\pi\cdot(N-1)/N} & e^{-j2\pi\cdot 2(N-1)/N} & \cdots & e^{-j2\pi\cdot(N-1)^2/N} \end{bmatrix} \cdot \begin{bmatrix} z(0) \\ z(1) \\ z(2) \\ \vdots \\ z(N-1) \end{bmatrix} = \begin{bmatrix} \underline{a}_0^{*T} \\ \underline{a}_1^{*T} \\ \vdots \\ \underline{a}_{N-1}^{*T} \end{bmatrix} \cdot \underline{z}$$

$$\begin{bmatrix} z(0) \\ z(1) \\ z(2) \\ \vdots \\ z(N-1) \end{bmatrix} = \frac{1}{\sqrt{N}} \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & e^{j2\pi/N} & e^{j2\pi\cdot 2/N} & \cdots & e^{j2\pi\cdot(N-1)/N} \\ 1 & e^{j2\pi\cdot 2/N} & e^{j2\pi\cdot 4/N} & \cdots & e^{j2\pi\cdot 2(N-1)/N} \\ \vdots & \vdots & & & \vdots \\ 1 & e^{j2\pi\cdot(N-1)/N} & e^{j2\pi\cdot 2(N-1)/N} & \cdots & e^{j2\pi\cdot(N-1)^2/N} \end{bmatrix} \cdot \begin{bmatrix} Z(0) \\ Z(1) \\ Z(2) \\ \vdots \\ Z(N-1) \end{bmatrix} = [\underline{a}_0 \quad \underline{a}_1 \quad \cdots \quad \underline{a}_{N-1}] \cdot \begin{bmatrix} Z(0) \\ Z(1) \\ \vdots \\ Z(N-1) \end{bmatrix}$$

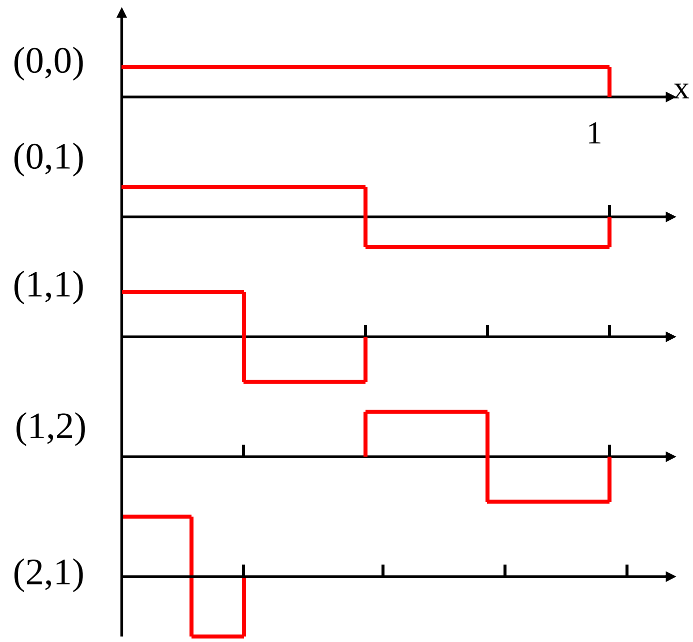
From Ken Lam's DCT talk 2001 (HK Polytech)

Example of 1-D DCT: N = 8



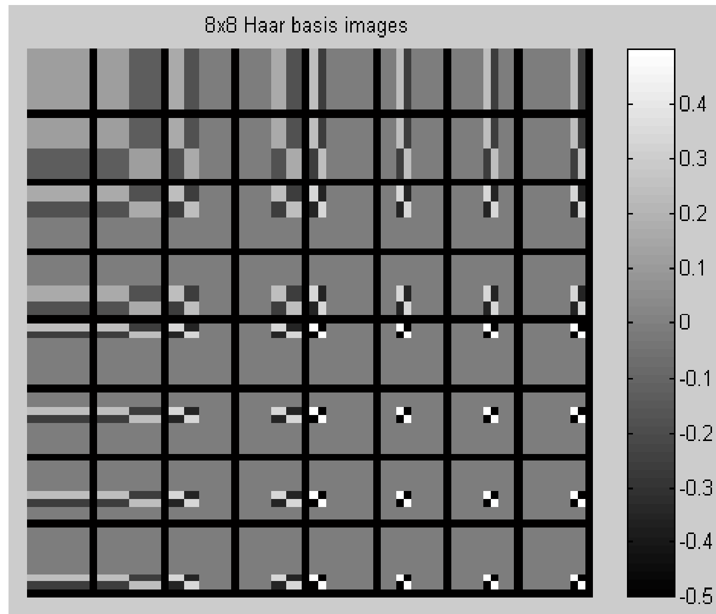
Haar Transform

- Haar basis functions: index by (p, q)
  - Scaling captures info. at different freq.
  - Translation captures info. at different locations
  - Transition at each scale *p* is localized according to *q*

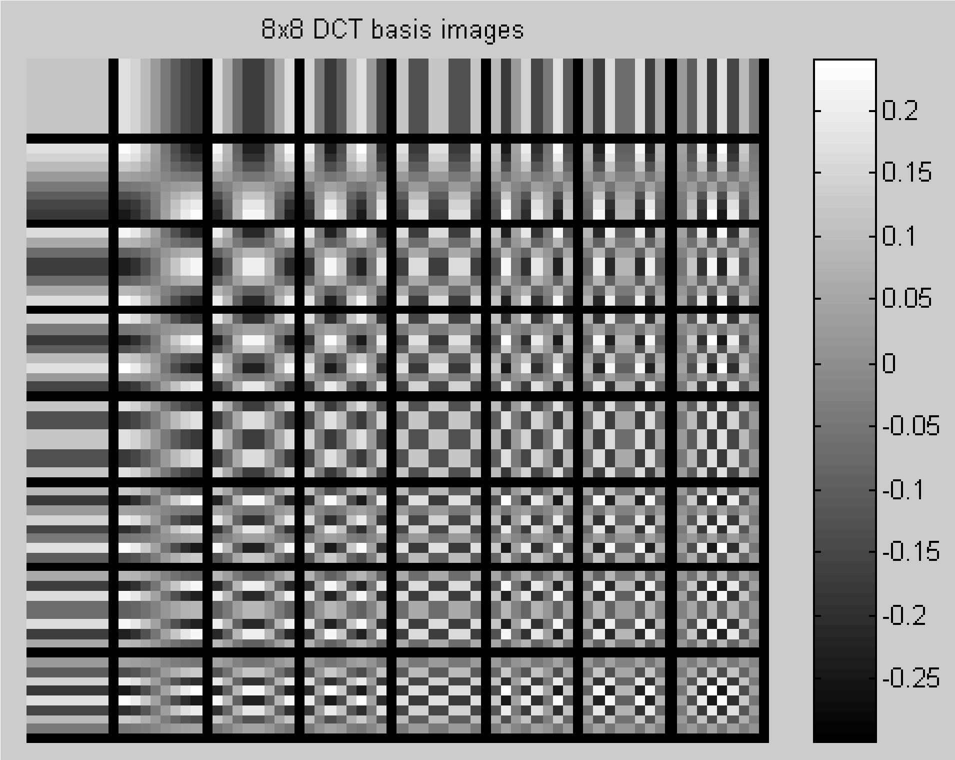


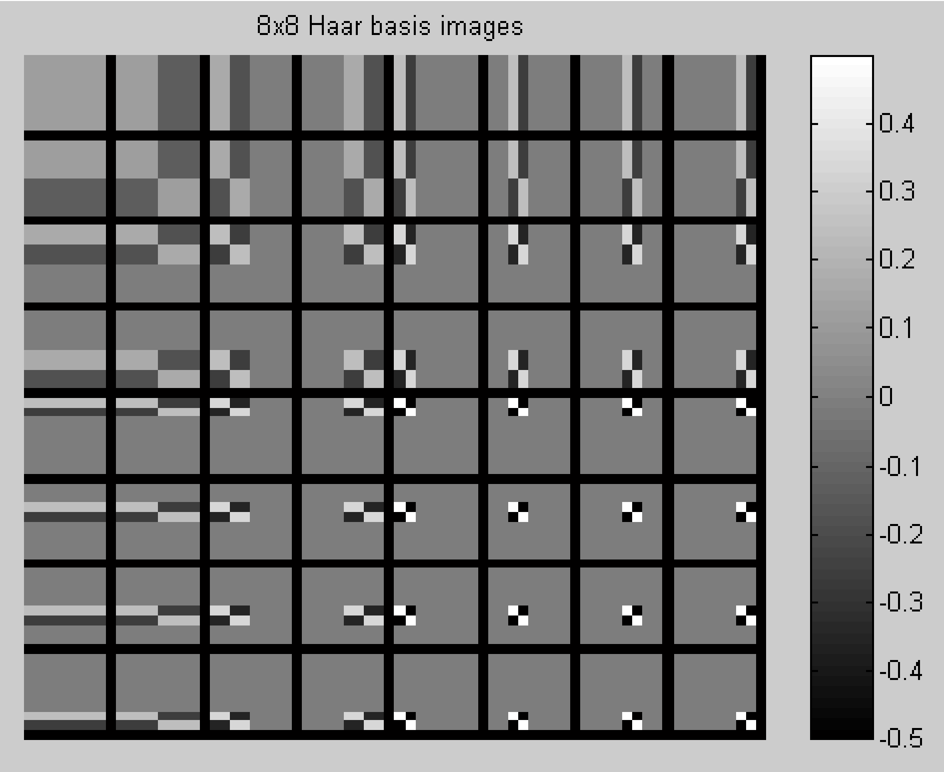
- Haar transform *H* ~ orthogonal
  - Sample Haar function to obtain transform matrix
  - Filter bank representation
    - *filtering and downsampling*
  - Relatively poor energy compaction
    - *Equiv. filter response doesn't have good cutoff & stopband attenuation*

=> Basis images of 2-D Haar transform



Compare Basis Images of DCT and Haar





See also: Jain's Fig.5.2 pp136

Compressive Sensing

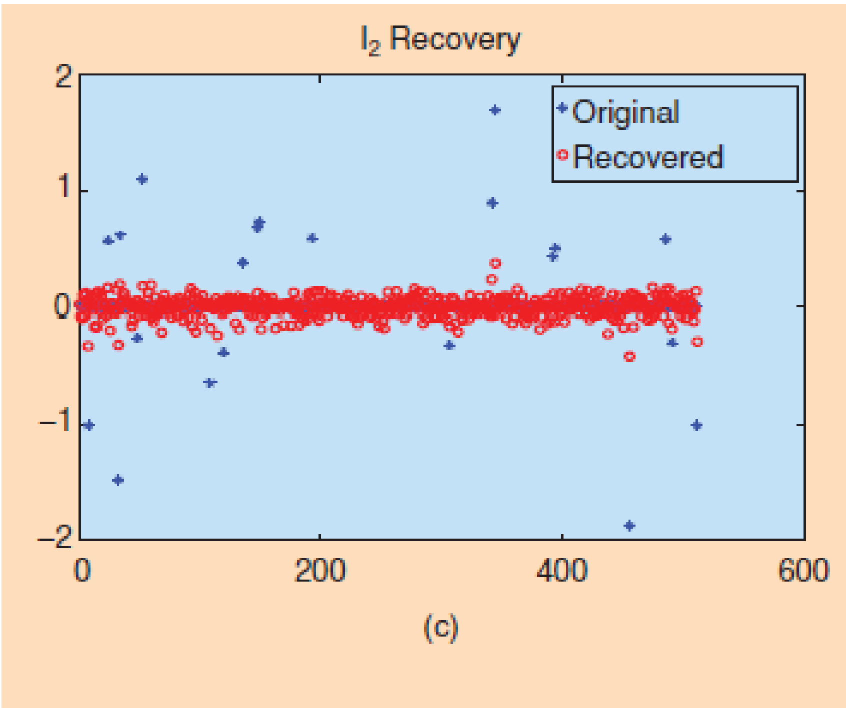
- Downsampling as a data compression tool
  - For bandlimited signals. Considered uniform sampling so far
- More general case of "sparsity" in some domain

  E.g. non-zero coeff. at a small # of frequencies but over a broad support of frequency?

  - How to leverage such sparsity to get reduced average sampling rate?
  - Can we sample at non-equally spaced intervals?
  - How to deal with real-world issues e.g. approx. but not exactly sparse?

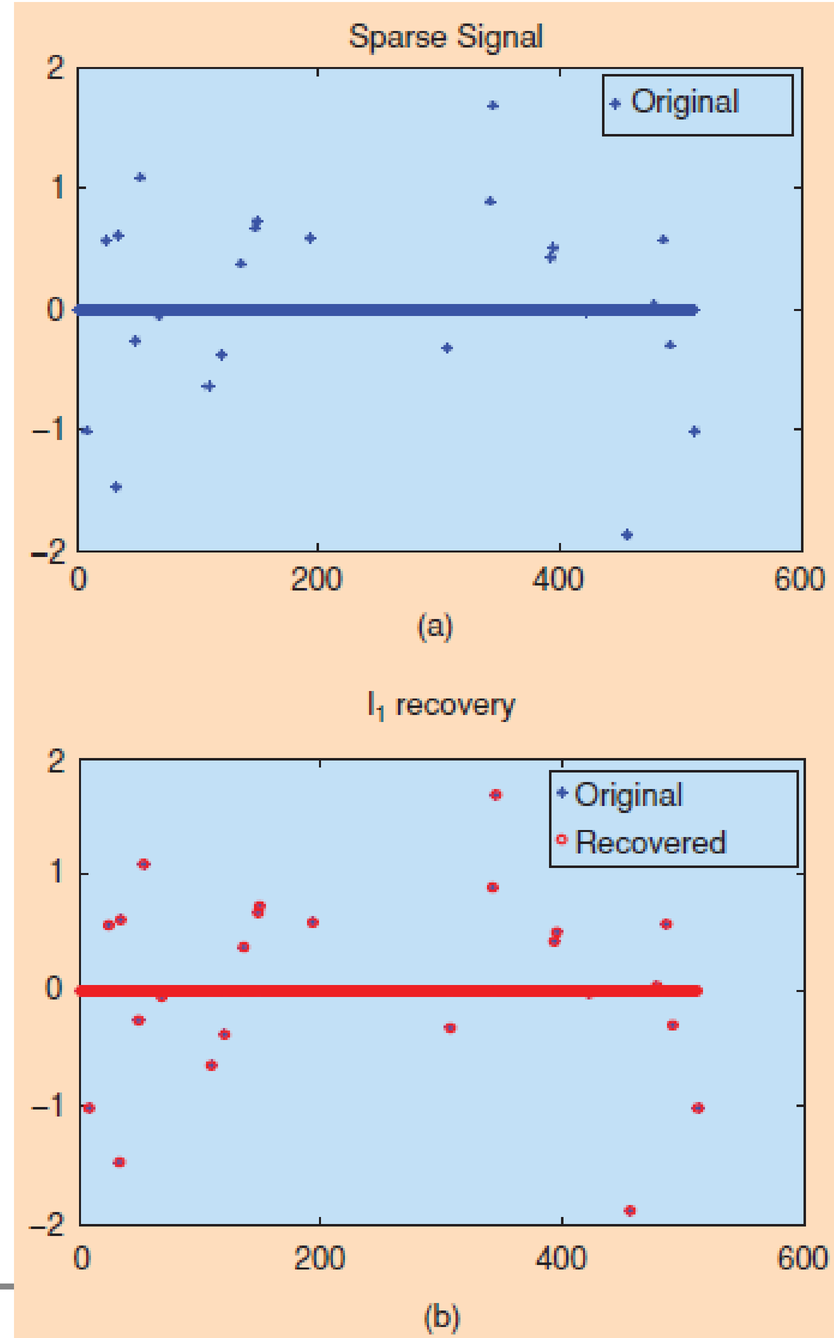
Ref: IEEE Signal Processing Magazine: Lecture Notes on Compressive Sensing (2007); Special Issue on Compressive Sensing (2008);
ENEE698A Fall 2008 Graduate Seminar: http://terpconnect.umd.edu/~dikpal/enee698a.html

L1 vs. L2 Optimization for Sparse Signal



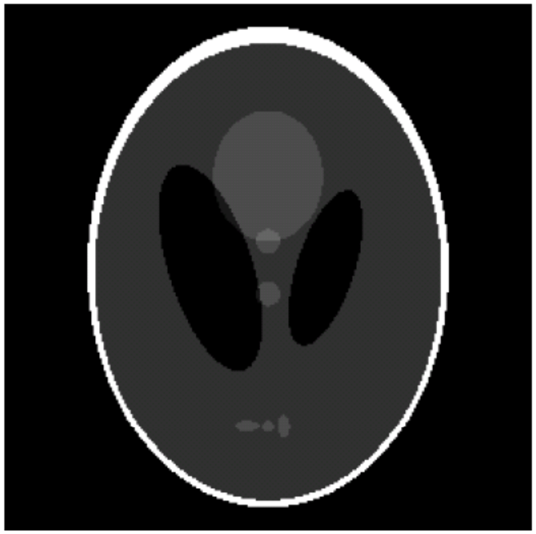
[FIG2] (a) A sparse real valued signal and (b) its reconstruction from 60 (complex valued) Fourier coefficients by ℓ_1 minimization. The reconstruction is exact. (c) The minimum energy reconstruction obtained by substituting the ℓ_1 norm with the ℓ_2 norm; ℓ_1 and ℓ_2 give wildly different answers. The ℓ_2 solution does not provide a reasonable approximation to the original signal.

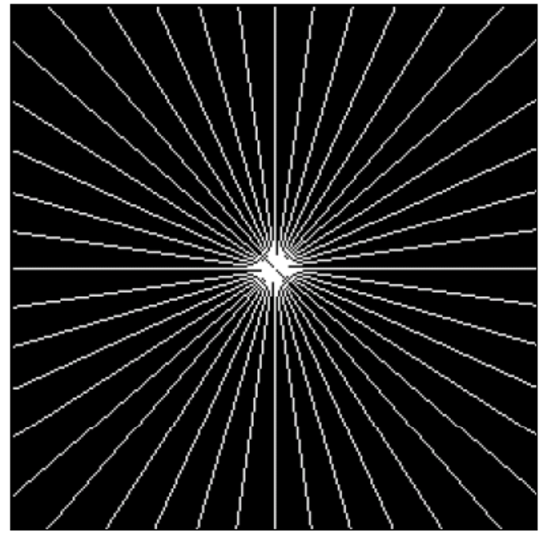
(Fig. from Candes-Wakins SPM'08 article)



Example: Tomography problem

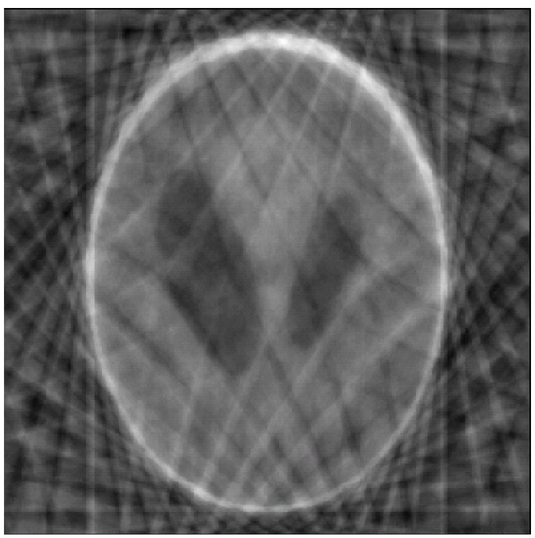
Logan-Shepp phantom
test image





Sampling in the frequency plane
Along 22 radial lines with 512
samples on each

Minimum energy
reconstruction



Reconstruction by minimizing
total variation

Slide source: by Dikpal Reddy, ENEE698A, http://terpconnect.umd.edu/~dikpal/enee698a.html

A Close Look at Wavelet Transform

Haar Transform – unitary

Orthonormal Wavelet Filters

Biorthogonal Wavelet Filters

Construction of Haar Functions

- Unique decomposition of integer $k \Leftrightarrow (p, q)$
  - $k = 0, \ldots, N-1$ with $N = 2^n$, $0 <= p <= n-1$
  - $q = 0, 1$ (for $p=0$); $1 <= q <= 2^p$ (for $p>0$)

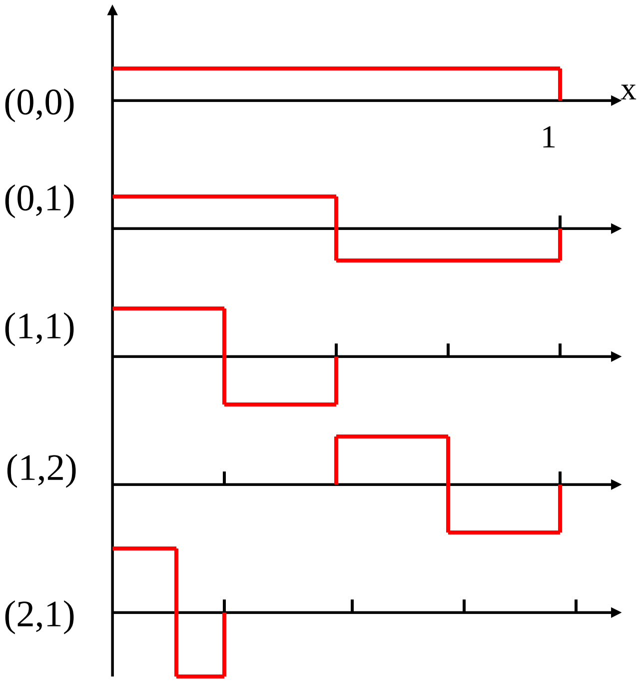
$$k = \underbrace{2^p}_{\text{power of 2}} + \underbrace{q - 1}_{\text{“remainder”}}$$

e.g.,	k=0	k=1	k=2	k=3	k=4	…
	(0,0)	(0,1)	(1,1)	(1,2)	(2,1)	…

- $h_k(x) = h_{p,q}(x)$ for $x \in [0,1]$

$$h_0(x) = h_{0,0}(x) = \frac{1}{\sqrt{N}} \quad \text{for } x \in [0,1]$$

$$h_k(x) = h_{p,q}(x) = \begin{cases} \frac{1}{\sqrt{N}} 2^{p/2} & \text{for } \frac{q-1}{2^p} \le x < \frac{q-\frac{1}{2}}{2^p} \\ -\frac{1}{\sqrt{N}} 2^{p/2} & \text{for } \frac{q-\frac{1}{2}}{2^p} \le x < \frac{q}{2^p} \\ 0 & \text{for other } x \in [0,1] \end{cases}$$



More on Wavelets (1)

- Linear expansion of a function via an expansion set
  - Form basis functions if the expansion is unique
- Orthogonal basis
- Non-orthogonal basis
  - Coefficients are computed with a set of dual-basis
- Discrete Wavelet Transform
  - Wavelet expansion gives a set of 2-parameter basis functions and expansion coefficients: scale and translation

$$f(t) = \sum_{l} a_l \psi_l(t)$$

orthogonal basis:

$$\langle \psi_k(t), \psi_l(t) \rangle = 0 \quad \text{for } k \neq l$$

$$a_k = \langle f(t), \psi_k(t) \rangle = \int f(t) \psi_k(t)\, dt$$

Dual basis set:

$$a_k = \langle f(t), \tilde{\psi}_k(t) \rangle$$

Wavelet expansion

$$f(t) = \sum_{k} \sum_{j} a_{j,k} \psi_{j,k}(t)$$

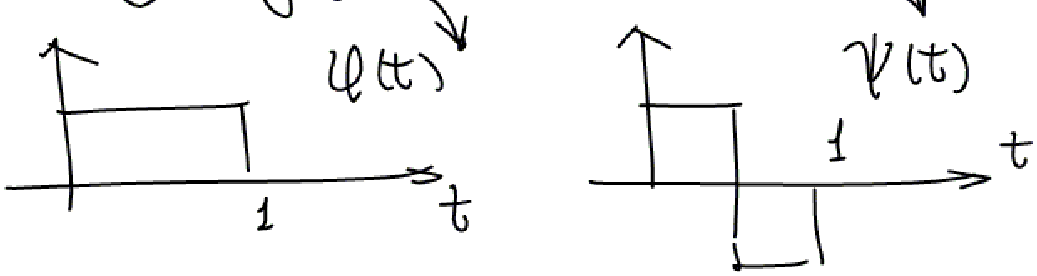
More on Wavelets (2)

- 1st generation wavelet systems:
  - Scaling and translation of a generating wavelet ("mother wavelet")
- Multiresolution conditions:
  - Use a set of basic expansion signals with half width and translated in half step size to represent a larger class of signals than the original expansion set (the "scaling function")
- Represent a signal by combining scaling functions and wavelets

$$\psi_{j,k}(t) = 2^{j/2}\psi(2^j t - k) \; ; \; \{\varphi(t-k)\} \to \{\varphi(2t-k)\}$$

$$f(t) = \sum_{k=-\infty}^{+\infty} c_k \varphi(t-k) + \sum_{k=-\infty}^{+\infty}\sum_{j=0}^{+\infty} d_{j,k}\,\psi(2^j t - k)$$

Haar scaling func and wavelets



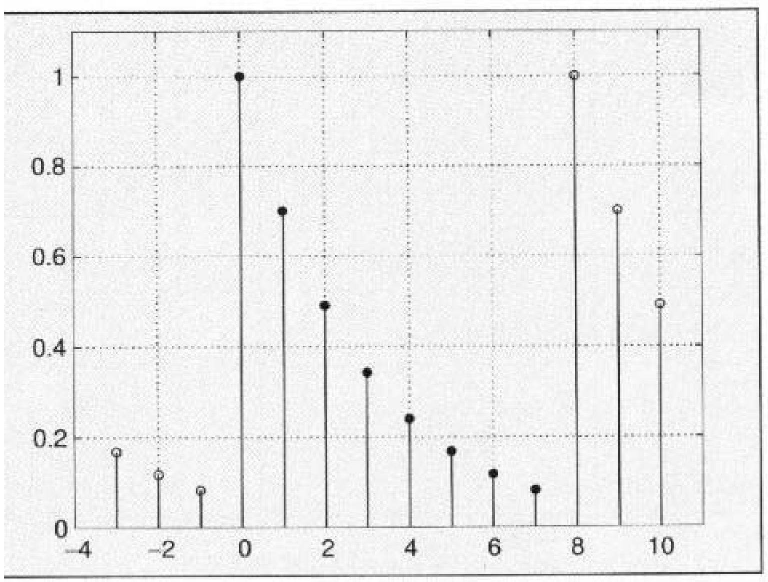
Orthonormal Filters

- Equiv. to projecting input signal to orthonormal basis
- Energy preservation property
  - Convenient for quantizer design
    - *MSE by transform domain quantizer is same as reconstruction MSE in image domain*
- Shortcomings: "coefficient expansion"
  - Linear filtering with N-element input & M-element filter
    ➔ (N+M-1)-element output ➔ (N+M)/2 after downsample
  - Length of output per stage grows ~ undesirable for compression
- Solutions to coefficient expansion
  - Symmetrically extended input (circular convolution) & Symmetric filter

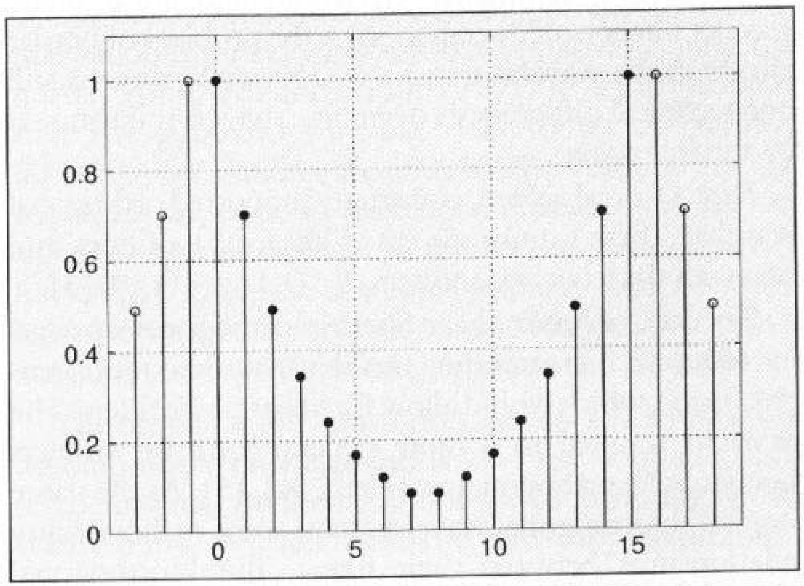
Solutions to Coefficient Expansion

- Circular convolution in place of linear convolution
  - Periodic extension of input signal
  - Problem: artifacts by large discontinuity at borders
- Symmetric extension of input
  - Reduce border artifacts (note the signal length doubled with symmetry)
  - Problem: output at each stage may not be symmetric

From Usevitch (IEEE Sig.Proc. Mag. 9/01)



9. The periodic extension of the input in Fig. 7.



▲ 11. Symmetric periodic extension of the original input shown in Fig. 7.

Solutions to Coefficient Expansion (cont'd)

- Symmetric extension + symmetric filters
  - No coefficient expansion and little artifacts
  - Symmetric filter (or asymmetric filter) => "linear phase filters" (*no phase distortion except by delays*)

- Problem
  - Only one set of linear phase filters for real FIR orthogonal wavelets
    ➔ Haar filters: (1, 1) & (1,-1)
    do not give good energy compaction

Ref: review ENEE630 discussions on FIR perfect reconstruction Qudrature Mirror Filters (QMF) for 2-channel filter banks.

From Usevitch (IEEE Sig.Proc. Mag. 9/01)

Biorthogonal Wavelets

- "Biorthogonal"
  - Basis in forward and inverse transf. are not the same but give overall perfect reconstruction (PR)
    - *recall EE630 PR filterbank*
  - No strict orthogonality for transf. filters so energy is not preserved
    - *But could be close to orthogonal filters' performance*
- Advantage
  - Covers a much broader class of filters
    - *including symmetric filters that eliminate coefficient expansion*
- Commonly used filters for compression
  - 9/7 biorthogonal symmetric filter
  - Efficient implementation: Lifting approach (ref: Swelden's tutorial)

Table 1. Two Sets of Linear Phase, Biorthogonal Wavelet Filter Coefficients.

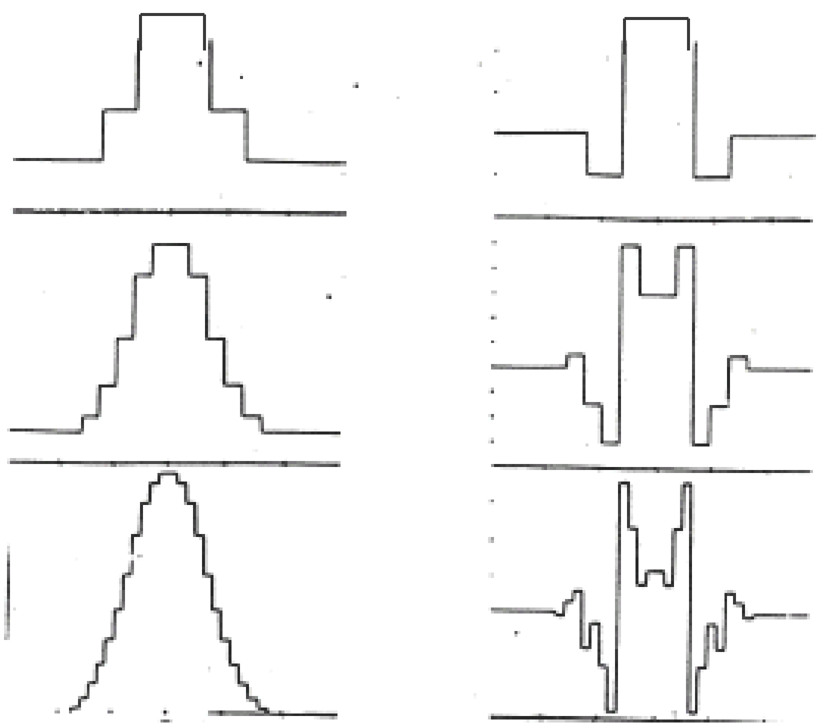
9/7 Filter Coefficients		5/3 Filter Coefficients		Filter Index
h_0	g_0	h_0	g_0	
0.852699	0.788486	1.060660	0.707107	0
0.377402	0.418092	0.353553	0.353553	−1, 1
−0.110624	−0.040689	−0.176777		−2, 2
−0.023849	−0.064539			−3, 3
0.037828				−4, 4

The 9/7 coefficients have the nice property that, although they are biorthogonal, they are very close to being orthogonal as shown in Table 2.

Smoothness Conditions on Wavelet Filter

- Ensure the low band coefficients obtained by recursive filtering can provide a smooth approximation of the original signal

$H_0(z) = G(z) \times G(z^2) \times G(z^4)$ in terms of freq. response $z = e^{j\omega}$: $G(\omega)\, G(2\omega)\, G(2^2\omega)$



From M. Vetterli's wavelet/filter-bank paper

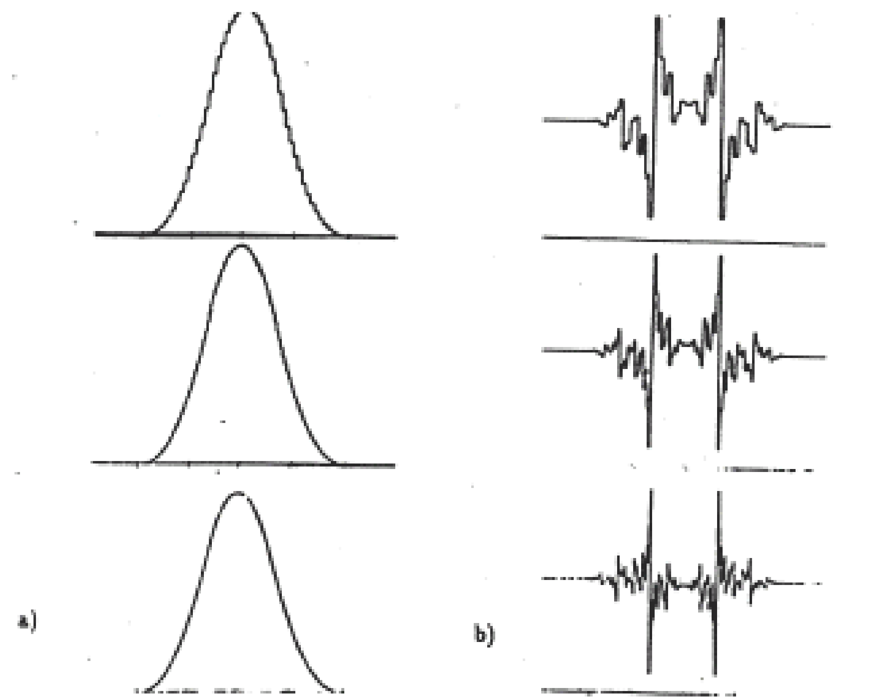


Figure 13: Iteration (28) for two simple filters. (a) [1,3,3,1] which converges to a continuous function. (b) [-1,3,3,-1] which converges to a discontinuous function.