Multi-rate Signal Processing
7. *M*-channel Maximally Decmiated Filter Banks

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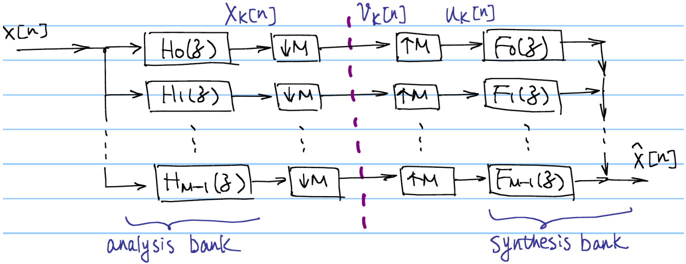
M-channel Maximally Decimated Filter Bank

M-ch. filter bank: To study more general conditions of alias-free & P.R.



As each filter has a passband of about $2\pi/M$ wide, the subband signal output can be decimated up to M without substantial aliasing.

The filter bank is said to be "maximally decimated" if this maximal decimation factor is used.

[Readings: Vaidynathan Book 5.4-5.5; Tutorial Sec.VIII]

The Reconstructed Signal and Errors Created

Relations between $\hat{X}(z)$ and $X(z)$: (details)

$$\hat{X}(z) = \sum_{l=0}^{M-1} A_\ell(z) X(W^\ell z)$$

- $A_\ell(z) \triangleq \frac{1}{M} \sum_{k=0}^{M-1} H_k(W^\ell z) F_k(z)$, $0 \leq \ell \leq M-1$.
- $X(W^\ell z)|_{z=e^{j\omega}} = X(\omega - \frac{2\pi\ell}{M})$, i.e., shifted version from $X(\omega)$.
- $X(W^\ell z)$: ℓ-th aliasing term, $A_\ell(z)$: gain for this aliasing term.

Conditions for LPTV, LTI, and PR

- In general, the M-channel filter bank is a LPTV system with period M.

- The aliasing term can be eliminated for every possible input $x[n]$ iff $A_\ell(z) = 0$ for $1 \leq \ell \leq M-1$. When aliasing is eliminated, the filter bank becomes an LTI system:

$$\hat{X}(z) = T(z)X(z),$$

where $T(z) \triangleq A_0(z) = \frac{1}{M}\sum_{\ell=0}^{M-1} H_k(z)F_k(z)$ is the overall transfer function, or distortion function.

- If $T(z) = cz^{-n_0}$, it is a perfect reconstruction system (i.e., free from aliasing, amplitude distortion, and phase distortion).

The Alias Component (AC) Matrix

From the definition of $A_\ell(z)$, we have in matrix-vector form:

$$M \underbrace{\begin{bmatrix} A_0(z) \\ A_1(z) \\ \vdots \\ A_{M-1}(z) \end{bmatrix}}_{\mathcal{A}(z)} = \underbrace{\begin{bmatrix} H_0(z) & H_1(z) & \cdots & H_{M-1}(z) \\ H_0(zW) & H_1(zW) & \cdots & H_{M-1}(zW) \\ \vdots & & & \\ H_0(zW^{M-1}) & H_1(zW^{M-1}) & \cdots & H_{M-1}(zW^{M-1}) \end{bmatrix}}_{\mathcal{H}(z)} \underbrace{\begin{bmatrix} F_0(z) \\ F_1(z) \\ \vdots \\ F_{M-1}(z) \end{bmatrix}}_{\mathbb{f}(z)}$$

$\mathcal{H}(z)$: $M \times M$ matrix called the "Alias Component matrix"

The condition for alias cancellation is

$$\mathcal{H}(z)\underline{\mathbb{f}}(z) = \underline{\mathbb{t}}(z), \quad \text{where } \underline{\mathbb{t}}(z) = \begin{bmatrix} MA_0(z) \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

The Alias Component (AC) Matrix

Now express the reconstructed signal as

$$\hat{X}(z) = \mathcal{A}^T(z)\mathcal{X}(z) = \frac{1}{M}\underline{\mathbb{f}}^T(z)\mathcal{H}^T(z)\mathcal{X}(z),$$

where $\mathcal{X}(z) = \begin{bmatrix} X(z) \\ X(zW) \\ \vdots \\ X(zW^{M-1}) \end{bmatrix}$.

Given a set of analysis filters $\{H_k(z)\}$, if $\det \mathcal{H}(z) \neq 0$, we can choose synthesis filters as $\underline{\mathbb{f}}(z) = \mathcal{H}^{-1}(z)\underline{\mathbb{t}}(z)$ to cancel aliasing and obtain P.R. by requiring

$$\underline{\mathbb{t}}(z) = \begin{bmatrix} cz^{-n_0} \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Difficulty with the Matrix Inversion Approach

$$\mathcal{H}^{-1}(z) = \frac{Adj[\mathcal{H}(z)]}{\det[\mathcal{H}(z)]}$$

- Synthesis filters $\{F_k(z)\}$ can be IIR even if $\{H_k(z)\}$ are all FIR.
- Difficult to ensure $\{F_k(z)\}$ stability (i.e. all poles inside the unit circle)
- $\{F_k(z)\}$ may have high order even if the order of $\{H_k(z)\}$ is moderate
- ......

$\Rightarrow$ Take a different approach for P.R. design via polyphase representation.

Type-1 PD for $H_k(z)$

Using Type-1 PD for $H_k(z)$:

$$H_k(z) = \sum_{\ell=0}^{M-1} z^{-\ell} E_{k\ell}(z^M)$$

We have

$$\underbrace{\begin{bmatrix} H_0(z) \\ \vdots \\ H_{M-1}(z) \end{bmatrix}}_{\underline{\mathbb{h}}(z)} = \underbrace{\begin{bmatrix} E_{00}(z^M) & E_{01}(z^M) & \cdots & E_{0,M-1}(z^M) \\ \vdots & & & \\ E_{M-1,0}(z^M) & - & - & E_{M-1,M-1}(z^M) \end{bmatrix}}_{\mathbb{E}(z^M)} \underbrace{\begin{bmatrix} 1 \\ z^{-1} \\ \vdots \\ z^{-(M-1)} \end{bmatrix}}_{\underline{e}(z)}$$

$$\Longleftrightarrow \quad \underline{\mathbb{h}}(z) = \mathbb{E}(z^M)\,\underline{e}(z)$$

$\mathbb{E}(z^M)$: $M \times M$ Type-1 polyphase component matrix for analysis bank

Type-2 PD for $F_k(z)$

Similarly, using Type-2 PD for $F_k(z)$:

$$F_k(z) = \sum_{\ell=0}^{M-1} z^{-(M-1-\ell)} R_{\ell k}(z^M)$$

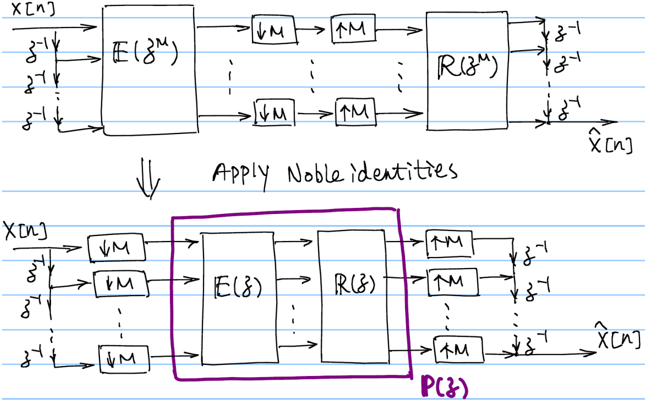
We have in matrix form:

$$\underbrace{[F_0(z) \; \cdots \; F_{M-1}(z)]} = \underbrace{[z^{-(M-1)}, z^{-(M-2)}, \cdots, 1]} \underbrace{\begin{bmatrix} R_{00}(z^M) & \cdots & R_{0,M-1}(z^M) \\ R_{10}(z^M) & \cdots & R_{1,M-1}(z^M) \\ \vdots & & \\ R_{M-1,0}(z^M) & \cdots & R_{M-1,M-1}(z^M) \end{bmatrix}}$$

$$\Longleftrightarrow \underline{\mathbb{f}}^T(z) = \underline{\mathbb{e}}_B^T(z) \mathbb{R}(z^M)$$

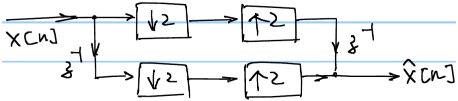
$\underline{\mathbb{e}}_B^T(z)$: reversely ordered version of $\underline{\mathbb{e}}(z)$
$\mathbb{R}(z^M)$: Type-2 polyphase component matrix for synthesis bank

Overall Polyphase Presentation



Combine polyphase matrices into one matrix: $\mathbb{P}(z) = \underbrace{\mathbb{R}(z)\mathbb{E}(z)}_{\text{note the order!}}$

Simple FIR P.R. Systems

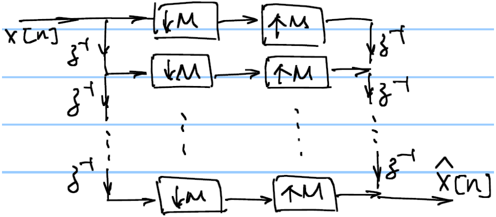


$\hat{X}(z) = z^{-1}X(z)$,
i.e., transfer function $T(z) = z^{-1}$

Extend to M channels:

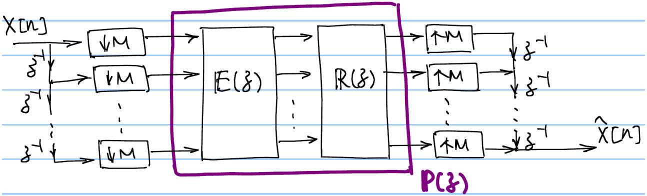
$H_k(z) = z^{-k}$
$F_k(z) = z^{-M+k+1}, 0 \leq k \leq M-1$
$\Rightarrow \hat{\mathbb{X}}(z) = z^{-(M-1)}\mathbb{X}(z)$

i.e. demultiplex then multiplex again



General P.R. Systems

Recall the polyphase implementation of M-channel filter bank:



Combine polyphase matrices into one matrix: $\mathbb{P}(z) = \mathbb{R}(z)\mathbb{E}(z)$

If $\mathbb{P}(z) = \mathbb{R}(z)\mathbb{E}(z) = \mathbb{I}$, then the system is equivalent to the simple system $\Rightarrow H_k(z) = z^{-k}$, $F_k(z) = z^{-M+k+1}$

In practice, we can allow $\mathbb{P}(z)$ to have some constant delay, i.e., $\mathbb{P}(z) = cz^{-m_0}\mathbb{I}$, thus $T(z) = cz^{-(Mm_0+M-1)}$.

Dealing with Matrix Inversion

To satisfy $\mathbb{P}(z) = \mathbb{R}(z)\mathbb{E}(z) = \mathbb{I}$, it seems we have to do matrix inversion for getting the synthesis filters $\mathbb{R}(z) = (\mathbb{E}(z))^{-1}$.

Question: Does this get back to the same inversion problem we have with the viewpoint of the AC matrix $\underline{\mathbb{f}}(z) = \mathcal{H}^{-1}(z)\underline{\mathbb{t}}(z)$?

Solution:

- $\mathbb{E}(z)$ is a physical matrix that each entry can be controlled.
  In contrast, for $\mathcal{H}(z)$, only 1st row can be controlled (thus hard to ensure desired $H_k(z)$ responses **and** $\underline{\mathbb{f}}(z)$ stability)
- We can choose FIR $\mathbb{E}(z)$ s.t. $\det \mathbb{E}(z) = \alpha z^{-k}$ thus $\mathbb{R}(z)$ can be FIR (and has determinant of similar form).

Summary: With polyphase representation, we can choose $\mathbb{E}(z)$ to produce desired $H_k(z)$ and lead to simple $\mathbb{R}(z)$ s.t. $\mathbb{P}(z) = cz^{-k}\mathbb{I}$.

Paraunitary

A more general way to address the need of matrix inversion:

Constrain $\mathbb{E}(z)$ to be **paraunitary**: $\tilde{\mathbb{E}}(z)\mathbb{E}(z) = d\mathbb{I}$

Here $\tilde{\mathbb{E}}(z) = \mathbb{E}_*^T(z^{-1})$, i.e. taking conjugate of the transfer function coeff., replace z with z^{-1} that corresponds to time reversely order the filter coeff., and transpose.

For further exploration: PPV Book Chapter 6.

Relation b/w Polyphase Matrix $\mathbb{E}(z)$ and AC Matrix $\mathcal{H}(z)$

The relation between $\mathbb{E}(z)$ and $\mathcal{H}(z)$ can be shown as:

$$\mathcal{H}(z) = [\mathbb{W}^*]^T \, \mathbb{D}(z) \quad \mathbb{E}^T(z^M)$$

where $\mathbb{W}$ is the $M \times M$ DFT matrix, and a diagonal delay matrix

$$\mathbb{D}(z) = \begin{bmatrix} 1 & & & \\ & z^{-1} & & \\ & & \ddots & \\ & & & z^{-(M-1)} \end{bmatrix}$$

(details) See also the homework.

Detailed Derivations

The Reconstructed Signal and Errors Created

① $X_k(z) = H_k(z)X(z)$

② $V_k(z) = \frac{1}{M}\sum_{\ell=0}^{M-1} \underbrace{H_k(W_M^{\ell} z^{1/M})X(W_M^{\ell} z^{1/M})}_{X_k(W_M^{\ell} z^{1/M})}$ where $W_M = e^{-j2\pi/M}$

③ $U_k(z) = V_k(z^M) = \frac{1}{M}\sum_{\ell=0}^{M-1} H_k(W_M^{\ell} z)X(W_M^{\ell} z)$

④ $\hat{X}(z) = \sum_{k=0}^{M-1} F_k(z)U_k(z)$

$= \sum_{\ell=0}^{M-1}\left[\frac{1}{M}\sum_{k=0}^{M-1} H_k(W^{\ell}z)F_k(z)\right]X(W^{\ell}z)$

$= \sum_{\ell=0}^{M-1} A_{\ell}(z)\, X(W^{\ell}z)$

- $A_\ell(z) \triangleq \frac{1}{M}\sum_{k=0}^{M-1} H_k(W^\ell z)F_k(z)$, $0 \le \ell \le M-1$.
- $\mathbb{X}(W^\ell z)|_{z=e^{j\omega}} = \mathbb{X}(\omega - \frac{2\pi\ell}{M})$, i.e., shifted version from $\mathbb{X}(\omega)$.
- $\mathbb{X}(W^\ell z)$: ℓ-th aliasing term, $A_\ell(z)$: gain for this aliasing term.

Review: Matrix Inversion

$$\mathcal{H}^{-1}(z) = \frac{Adj[\mathcal{H}(z)]}{\det[\mathcal{H}(z)]}$$

Adjugate or classical adjoint of a matrix:

$\{Adj[\mathcal{H}(z)]\}_{ij} = (-1)^{i+j} M_{ji}$

where M_{ji} is the (j, i) minor of $\mathcal{H}(z)$ defined as the determinant of the matrix by deleting the j-th row and i-th column.

An Example of P.R. Systems

$$H_0(z) = 2 + z^{-1},\ \ H_1(z) = 3 + 2z^{-1},$$

$$\mathbb{E}(z) = \left[\begin{array}{cc} 2 & 1 \\ 3 & 2 \end{array}\right],\ \mathbb{E}^{-1}(z) = \frac{\text{Adj } \mathbb{E}(z)}{\det \mathbb{E}(z)} = 1 \times \left[\begin{array}{cc} 2 & -1 \\ -3 & 2 \end{array}\right].$$

Choose $\mathbb{R}(z) = \mathbb{E}^{-1}(z)$ s.t. $\mathbb{P}(z) = \mathbb{R}(z)\mathbb{E}(z) = \mathbb{I}$,

$$\therefore \mathbb{R}(z) = \left[\begin{array}{cc} 2 & -1 \\ -3 & 2 \end{array}\right]$$

$$\left[\begin{array}{cc} F_0(z) & F_1(z) \end{array}\right] = \left[\begin{array}{cc} z^{-1} & 1 \end{array}\right] \mathbb{R}(z^2) = \left[\begin{array}{cc} 2z^{-1} - 3, & -z^{-1} + 2 \end{array}\right]$$

$$\Rightarrow \begin{cases} F_0(z) = -3 + 2z^{-1} \\ F_1(z) = 2 - z^{-1} \end{cases}$$