Multi-rate Signal Processing
3. The Polyphase Representation

Electrical & Computer Engineering
University of Maryland, College Park

Acknowledgment: ENEE630 slides were based on class notes developed by Profs. K.J. Ray Liu and Min Wu. The LaTeX slides were made by Prof. Min Wu and Mr. Wei-Hong Chuang.

Contact: *minwu@umd.edu*. Updated: September 16, 2012.

Polyphase Representation: Basic Idea

Example: FIR filter $H(z) = 1 + 2z^{-1} + 3z^{-2} + 4z^{-3}$

Group even and odd indexed coefficients, respectively:
$\Rightarrow H(z) = (1 + 3z^{-2}) + z^{-1}(2 + 4z^{-2})$,

More generally: Given a filter $H(z) = \sum_{n=-\infty}^{\infty} h[n]z^{-n}$, by grouping the odd and even numbered coefficients, we can write

$$H(z) = \sum_{n=-\infty}^{\infty} h[2n]z^{-2n} + z^{-1} \sum_{n=-\infty}^{\infty} h[2n+1]z^{-2n}$$

Polyphase Representation: Definition

$$H(z) = \sum_{n=-\infty}^{\infty} h[2n]z^{-2n} + z^{-1} \sum_{n=-\infty}^{\infty} h[2n+1]z^{-2n}$$

Define $E_0(z)$ and $E_1(z)$ as two polyphase components of $H(z)$:

$$E_0(z) = \sum_{n=-\infty}^{\infty} h[2n]z^{-n},$$
$$E_1(z) = \sum_{n=-\infty}^{\infty} h[2n+1]z^{-n},$$

We have

$$H(z) = E_0(z^2) + z^{-1}E_1(z^2)$$

- These representations hold whether $H(z)$ is FIR or IIR, causal or non-causal.
- The polyphase decomposition can be applied to any sequence, not just impulse response.

FIR and IIR Example

1. FIR filter: $H(z) = 1 + 2z^{-1} + 3z^{-2} + 4z^{-3}$

   $\because H(z) = (1 + 3z^{-2}) + z^{-1}(2 + 4z^{-2})$,

   $\therefore E_0(z) = 1 + 3z^{-1}; \quad E_1(z) = 2 + 4z^{-1}$

2. IIR filter: $H(z) = \frac{1}{1-\alpha z^{-1}}$.

   Write into the form of $H(z) = E_0(z^2) + z^{-1}E_1(z^2)$:

   $$\because H(z) = \frac{1}{1-\alpha z^{-1}} \times \frac{1+\alpha z^{-1}}{1+\alpha z^{-1}} = \frac{1+\alpha z^{-1}}{1-\alpha^2 z^{-2}}$$

   $$= \frac{1}{1-\alpha^2 z^{-2}} + z^{-1}\frac{\alpha}{1-\alpha^{-2} z^{-2}}$$

   $$\therefore E_0(z) = \frac{1}{1-\alpha^2 z^{-1}}; \quad E_1(z) = \frac{\alpha}{1-\alpha^{-2} z^{-1}}$$

(For higher order filters: first write in the sum of 1st order terms)

Extension to M Polyphase Components

For a given integer M and $H(z) = \sum_{n=-\infty}^{\infty} h[n]z^{-n}$, we have:

$$H(z) = \sum_{n=-\infty}^{\infty} h[nM]z^{-nM} + z^{-1} \sum_{n=-\infty}^{\infty} h[nM+1]z^{-nM} \\ + \quad \ldots \quad + z^{-(M-1)} \sum_{n=-\infty}^{\infty} h[nM+M-1]z^{-nM}$$

Type-1 Polyphase Representation

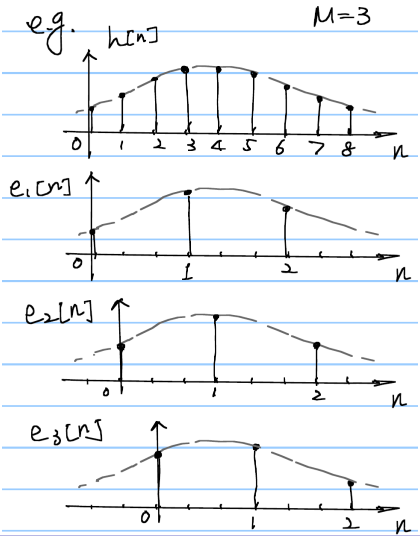
$$H(z) = \sum_{\ell=0}^{M-1} z^{-\ell} E_\ell(z^M)$$

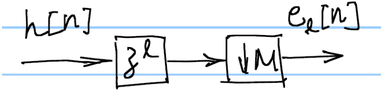
where the ℓ-th polyphase components of $H(z)$ given M is

$$E_\ell(z) \triangleq \sum_{n=-\infty}^{\infty} e_\ell[n]z^{-n} = \sum_{n=-\infty}^{\infty} h[nM+\ell]z^{-n}$$

Note: $0 \le \ell \le (M-1)$; strictly we may denote as $E_\ell^{(M)}(z)$.

Example: $M = 3$





z^{ℓ}: time advance

(there is a delay term when putting together the polyphase components)

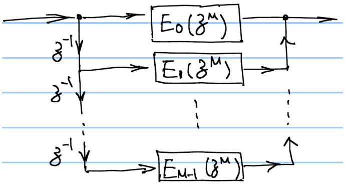
Alternative Polyphase Representation

If we define $R_\ell(z) = E_{M-1-\ell}(z),\ 0 \le \ell \le M-1$, we arrive at the

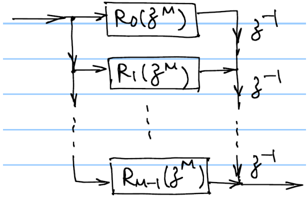
Type-2 polyphase representation

$$H(z) = \sum_{\ell=0}^{M-1} z^{-(M-1-\ell)} R_\ell(z^M)$$

Type-1: $E_k(z)$ is ordered consistently with the number of delays in the input



Type-2: reversely order the filter $R_k(z)$ with respect to the delays



Issues with Direct Implementation of Decimation Filters

Decimation Filters: $\longrightarrow \boxed{H(z)} \longrightarrow \boxed{\downarrow M} \longrightarrow$

Question: Any wasteful effort in the direct implementation?

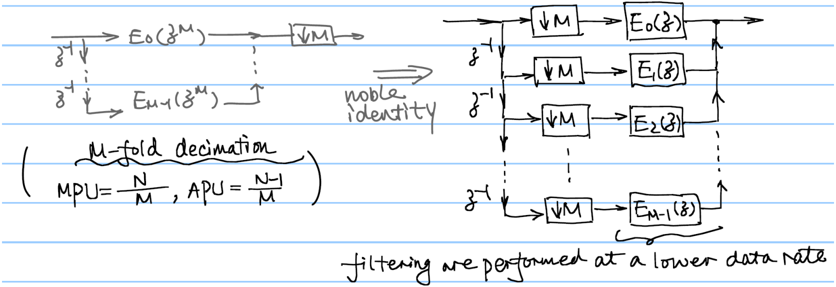
- The filtering is applied to all original signal samples, even though **only every** M **filtering output** is retained finally.
- Even if we let $H(z)$ operates only for time instants multiple of M and idle otherwise, all multipliers/adders have to produce results **within one step of time**.

- Can $\downarrow M$ be moved before $H(z)$?

  Only when $H(z)$ is a function of z^M, we can apply the noble identities to switch the order.

Efficient Structure for Decimation Filter

Apply Type-1 polyphase representation:



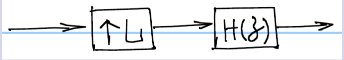
Computational Cost

For FIR filter $H(z)$ of length N:

- Total cost of N multipliers and $(N-1)$ adders is unchanged.
- Considering multiplications per input unit time (MPU) and additions per input unit time (APU),
  $E_k(z)$ now operates at a lower rate:
  only N/M **MPU and** $(N-1)/M$ **APU** are required.
- This is as opposed to N MPU and $(N-1)$ APU at every M instant of time and system idling at other instants, which leads to inefficient resource utilization.

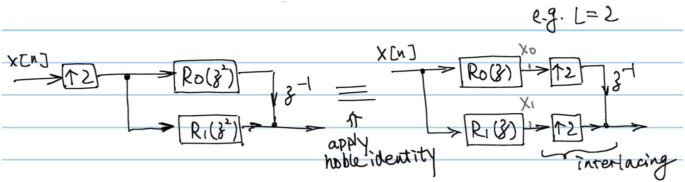
  (i.e., requires use fast additions and multiplications but use them only $1/M$ of time)

Polyphase for Interpolation Filters



Observe: the filter is applied to a signal at a high rate, even though many samples are zero when coming out of the expander.

Using the Type-2 polyphase decomposition:



$H(z) = z^{-1}R_0(z^2) + R_1(z^2)$:

- 2 polyphase components
- $R_k(z)$ is half length of $H(z)$

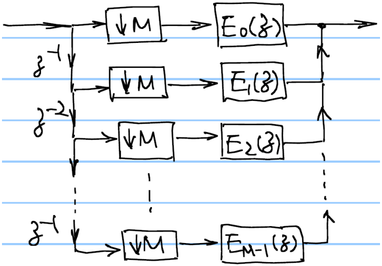
The complexity of the system is N MPU and $(N-2)$ APU.

General Cases

In general, for FIR filters with length N:

M-fold decimation:

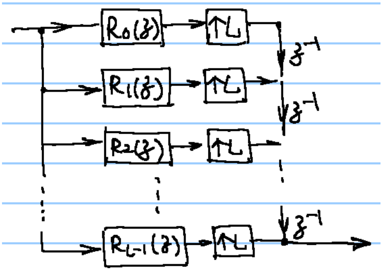
$$\text{MPU} = \frac{N}{M}, \ \text{APU} = \frac{N-1}{M}$$



filtering is performed at a lower data rate

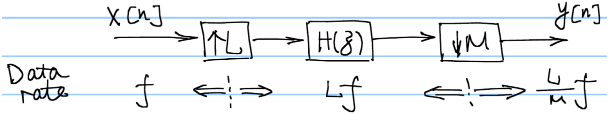
L-fold interpolation:

$$\text{MPU} = N, \ \text{APU} = N - L$$



$$\text{APU} = (\frac{N}{L} - 1) \times L$$

Fractional Rate Conversion

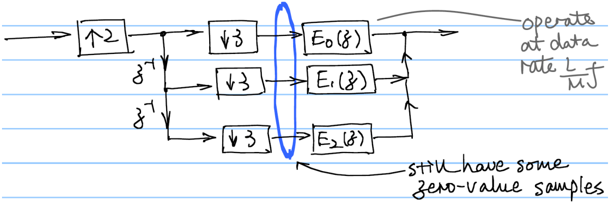


- Typically L and M should be chosen to have no common factors greater than 1 (o.w. it is wasteful as we make the rate higher than necessary only to reduce it down later)
- $H(z)$ filter needs to be fast as it operates in high data rate.
- The direct implementation of $H(z)$ is inefficient:

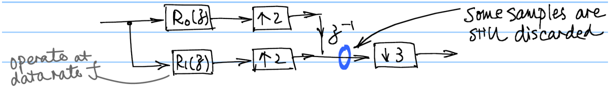
$$\begin{cases} \text{there are } L-1 \text{ zeros in between its input samples} \\ \text{only one out of } M \text{ samples is retained} \end{cases}$$

Example: $L = 2$ and $M = 3$

1. Use Type-1 polyphase decomposition (PD) for decimator:



2. Use Type-2 PD for interpolator:



Example: $L = 2$ and $M = 3$

3. Try to take advantage of both:

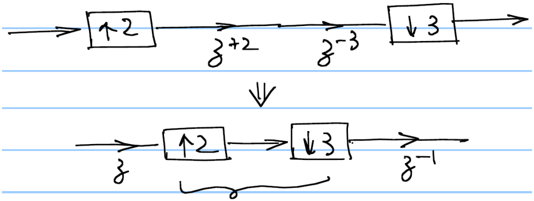
   Question: What's the lowest possible data rate to process?
   f/M

   Challenge: Can't move $\uparrow 2$ further to the right and $\downarrow 3$ to the left across the delay terms.

Trick to enable interchange of $\uparrow L$ and $\downarrow M$

$$z^{-1} = z^{-3} \cdot z^{2}$$

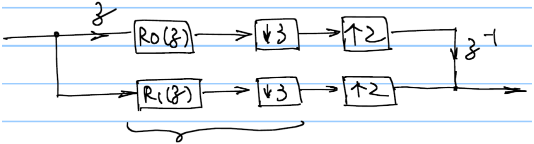
- z^{-3} and z^{2} can be considered as filters in z^{-M} and z^{+L}
- Noble identities can be applied:



can be interchanged as they are relatively prime

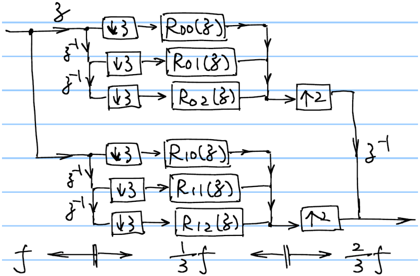
Overall Efficient Structure

Now it becomes



can move decimation earlier by Type-1 PD of $R_k(z)$

Finally,



$$R_0(z) = R_{00}(z^3) + z^{-1}R_{01}(z^3) + z^{-2}R_{02}(z^3)$$

$$R_1(z) = R_{10}(z^3) + z^{-1}R_{11}(z^3) + z^{-2}R_{12}(z^3)$$

Observations

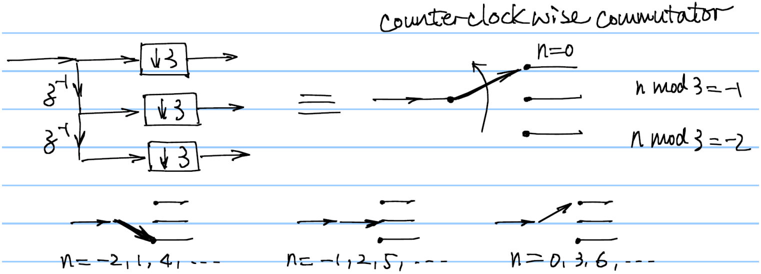
- For N-th order $H(z)$: MPU $= (N+1)/M \Rightarrow$ independent of L
- The final structure is the most efficient:
  $\begin{cases} \text{Decimators are moved to the left of all computational units} \\ \text{Expanders are moved to the right of all computational units} \end{cases}$
  Thus the computation is operated at the lowest possible rate.
- The above scheme works for arbitrary integers L and M as long as they are relatively prime.

  Under this condition, we have:
  1. $\exists n_0, n_1 \in \mathbb{Z}$ s.t. $n_1 M - n_0 L = 1$ (**Euclid's theorem**)
     We can then decompose $z^{-1} = z^{n_0 L} z^{-n_1 M}$
  2. $\uparrow L$ and $\downarrow M$ are interchangeable

Commutator Model: A Delay Chain followed by Decimators

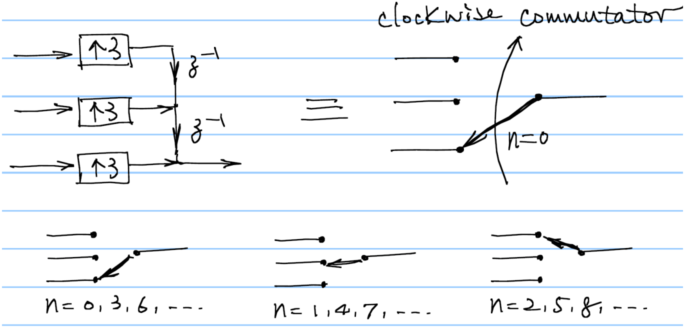
Polyphase implementation is often characterized by

1. A delay chain followed by a set of decimators,



Commutator Model: Expanders followed by A Delay Chain

2. A set of expanders followed by a delay chain

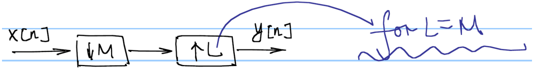


Commutator/switch model is an appealing conceptual tool to visualize these operations

Discussions: Linear Periodically Time Varying Systems

Some multirate systems that we have seen are linear periodically time varying (LPTV) systems.

e.g.,



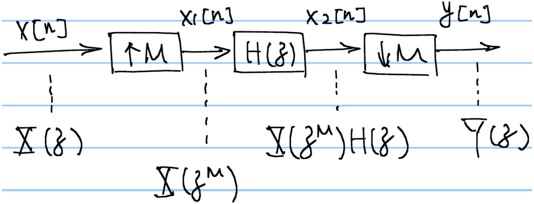
$$
\begin{aligned}
y[n] &= \begin{cases} x[n] & \text{if } n \text{ is multiple of } M \\ 0 & \textit{otherwise} \end{cases} \\
&= x[n] \cdot c[n]
\end{aligned}
$$

$c[n]$ is a comb function: takes 1 for n is multiple of M and 0 o.w.

$\Rightarrow$ This is a linear system with periodically time varying response coefficients, and the period is M.

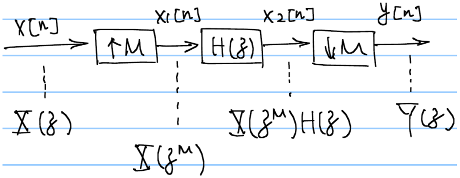
Time-invariant System with Decimator / Expander

Even though $\uparrow L$ and $\downarrow M$ are time-varying, a cascaded system having them as building blocks may become time-invariant.



This structure is the same as a fractional decimation system with $L = M$.

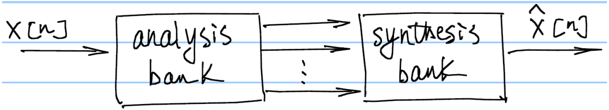
Time-invariant System with $\uparrow M$ & $\downarrow M$



details

Recall: $[\mathbb{X}(z)]_{\downarrow M} = \frac{1}{M}\sum_{k=0}^{M-1} \mathbb{X}\left(W_M^k z^{1/M}\right)$

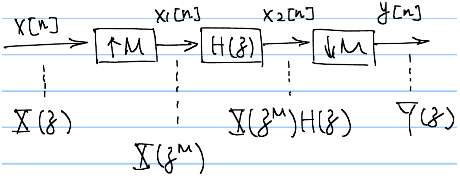
Perfect Reconstruction (PR) Systems



• The above system is said to be a **perfect reconstruction** system if $\hat{x}[n] = cx[n - n_0]$ for some $c \neq 0$ and integer n_0, i.e., the output is identical to the input, except a constant multiplicative factor and some fixed delay.

• Look ahead: we'll see the quadrature mirror filter bank (QMF) is generally a LPTV system, reduces to an LTI system when aliasing is completely cancelled, and achieves PR for certain analysis/synthesis filters.

Special Time-invariant System with $\uparrow M$ & $\downarrow M$

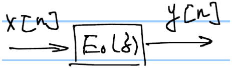


(back)

Recall: $[\mathbb{X}(z)]_{\downarrow M} = \frac{1}{M}\sum_{k=0}^{M-1}\mathbb{X}\left(W_M^k z^{1/M}\right)$

$$\begin{aligned}\mathbb{Y}(z) &= \left[\mathbb{X}(z^M)H(z)\right]_{\downarrow M} \\ &= \frac{1}{M}\sum_{k=0}^{M-1}\mathbb{X}\left(W_M^{Mk}z\right)H\left(W_M^k z^{1/M}\right) = \mathbb{X}(z)[H(z)]_{\downarrow M}\end{aligned}$$

$[H(z)]_{\downarrow M}$ implies decimating the impulse response $h[n]$ by M-fold, corresponding to the 0-th polyphase component of $H(z)$.

$\Rightarrow \mathbb{Y}(z) = \mathbb{X}(z)E_0(z)$, i.e., 

, an LTI system.