# Detailed Derivations

# Transform-Domain Analysis of Expanders

Z-Transform Relation between the Input and Output:

$$\mathbb{Y}_E(z) = \mathbb{X}(z^L)$$

Proof:

$$\mathbb{Y}_E(z) = \sum_{n=-\infty}^{+\infty} y_E[n] z^{-n} = \sum_{n=kL,\, k\in\mathbb{Z}} y_E[n] z^{-n}$$

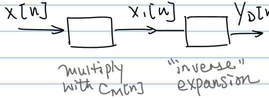
(i.e. exclude expanded zeros)

$$= \sum_{k=-\infty}^{+\infty} y_E[kL] z^{-kL}$$

$$= \sum_{k=-\infty}^{+\infty} x[k] (z^L)^{-k} = \mathbb{X}(z^L)$$

# Transform-Domain Analysis of Decimators

$$Y_D(z) = \sum_{n=-\infty}^{+\infty} y_D[n] z^{-n} = \sum_{n=-\infty}^{+\infty} x[nM] z^{-n}$$




Define $x_1[n] = \begin{cases} x[n] & \text{if } n \text{ is integer multiple of } M \\ 0 & \text{o.w.} \end{cases}$

We have $Y_D(z) = \sum_{k=nM,\, n\in\mathbb{Z}} x[k]\, z^{-k/M} = \sum_{k=-\infty}^{+\infty} x_1[k] \left(z^{1/M}\right)^{-k} = X_1(z^{1/M})$

change variable

# Transform-Domain Analysis of Decimators

To establish the transform-domain relation between $X_1(z)$ and $X(z)$:

We note $x_1[n]$ can be written as

$$x_1[n] = c_M[n]\, x[n]$$

where $c_M[n] = \begin{cases} 1 & \text{if } n \text{ is integer multiple of } M \\ 0 & \text{o.w.} \end{cases}$

("comb" sequence)

$\boxed{\text{Trick}}$ Using the $M^{th}$ root of unity $W_M$ defined as

$$W_M = e^{-j2\pi/M}$$

We have

$$c_M[n] = \frac{1}{M} \sum_{k=0}^{M-1} W_M^{-kn}$$

# Transform-Domain Analysis of Decimators

By the definition of ZT:

$$X_1(z) = \sum_{n=-\infty}^{+\infty} x_1[n]\, z^{-n} = \sum_{n=-\infty}^{+\infty} c_M[n]\, x[n]\, z^{-n}$$

$$= \sum_{n=-\infty}^{+\infty} \frac{1}{M} \sum_{k=0}^{M-1} W_M^{-kn}\, x[n]\, z^{-n}$$

$$= \frac{1}{M} \sum_{k=0}^{M-1} \sum_{n=-\infty}^{+\infty} x[n] \left(z \cdot W_M^{k}\right)^{-n}$$

$$= \frac{1}{M} \sum_{k=0}^{M-1} X\left(W_M^{k} \cdot z\right)$$

$$\therefore\ Y_D(z) = X_1\left(z^{1/M}\right) = \frac{1}{M} \sum_{k=0}^{M-1} X\left(W_M^{k}\, z^{1/M}\right)$$

$$= \frac{1}{M} \sum_{k=0}^{M-1} X\left(e^{-j2\pi k/M}\, z^{1/M}\right)$$

# Transform-Domain Analysis of Decimators

Fourier Spectrum: set $z = e^{j\omega}$

$$Y_D(e^{j\omega}) = \frac{1}{M} \sum_{k=0}^{M-1} X(e^{-j2\pi k/M} e^{j\omega/M})$$

$$= \frac{1}{M} \sum_{k=0}^{M-1} X(e^{j(\omega-2\pi k)/M})$$

$$\therefore \underbrace{Y_D(\omega) = \frac{1}{M} \sum_{k=0}^{M-1} X\left(\frac{\omega-2\pi k}{M}\right)}$$

# Time Domain Descriptions of Multirate Filters

Recall:

1. $x[n] \longrightarrow \boxed{h[n]} \xrightarrow{w[n]} \boxed{\downarrow M} \longrightarrow y[m]$

$$w[n] = \sum_k h[k]\, x[n-k] = \sum_k h[n-k]\, x[k]$$

$$y[m] = w[Mm] = \sum_k h[k]\, x[Mm-k]$$

$$= \sum_k \underline{h[Mm-k]}\, x[k]$$

2. $x[n] \longrightarrow \boxed{\uparrow L} \xrightarrow{w[m]} \boxed{h[n]} \longrightarrow y[m]$

$$w[m] = \begin{cases} x[m/L] & \text{if } m \text{ is multiple of } L \\ 0 & \text{o.w.} \end{cases}$$

$$y[m] = \sum_k w[k]\, h[m-k] = \sum_{k'} x[k']\, h[m-k'L]$$

only keep non-zero $w[m]$
for all $k = k'L$

# Input-Output Relation of DFT Filter Bank

$k^{th}$ row of $\overline{W}^*$

$$X_k[n] = \sum_{i=0}^{M-1} S_i[n] W^{-ki} \sim M\text{-point IDFT of } \{S_0[n], \cdots S_{M-1}[n]\}.$$

(except no factor $1/M$)

$$\mathcal{X}_k(z) = \sum_{i=0}^{M-1} S_i(z) W^{-ki} \xrightarrow{\text{delay}} = \sum_{i=0}^{M-1} z^{-i} W^{-ki} \mathcal{X}(z)$$

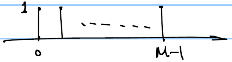
$$= \underbrace{\sum_{i=0}^{M-1} (W^k z)^{-i}}_{} \mathcal{X}(z)$$

Define this as $H_k(z)$, we have:

(the transfer functions)

# Relation between $H_i(z)$

$H_0(z) = \sum_{i=0}^{M-1} z^{-i}$ i.e. ZT of a rectangular window




$\xrightarrow{ZT} \quad H_0(z) = \dfrac{1 - z^{-M}}{1 - z^{-1}}$

$\longrightarrow \quad |H_0(\omega)| = \left| \dfrac{\sin(M\omega/2)}{\sin(\omega/2)} \right|$

$H_k(z) = H_0(W^k z) = H_0(e^{-j2\pi k/M} z)$

$\xrightarrow{z = e^{j\omega}} \quad H_k(\omega) = H_0\left(\omega - \dfrac{2\pi k}{M}\right)$

i.e. uniformly shifted version of $H_0(\omega)$ spectrum

# Time-domain Interpretation of the Uniform DFT FB

$$X_k[n+M-1] = \sum_{i=0}^{M-1} x[n+M-1-i]\, W^{-ki}$$

let $l = M-1-i$
$\Rightarrow i = M-1-l$

Here we consider a delayed version of x_k[ ] for convenience.

$$= \sum_{l=0}^{M-1} x[n+l]\, W^{kl} \cdot W^{-k(M-1)}$$

Note $W^M = 1$

$$= \underbrace{W^{k}}_{\text{phase shift}} \sum_{l=0}^{M-1} x[n+l]\, W^{kl}$$

(linear phase term reflects time delay: $e^{-j\omega}\big|_{\omega=\frac{2\pi}{M}k}$)

$k^{th}$ point of the M-point DFT of $\{x[n], \cdots, x[n+M-1]\}$

# Condition for $y_1[n] = y_2[n]$

Examine the ZT of $y_1[n]$ and $y_2[n]$:

$$\left.\begin{aligned} Y_1(z) &= X_1(z^L) \\ X_1(z) &= \frac{1}{M}\sum_{k=0}^{M-1} X(W_M^k z^{1/M}) \end{aligned}\right\} \Rightarrow Y_1(z) = \frac{1}{M}\sum_{k=0}^{M-1} X(W_M^k z^{L/M})$$

$$\left.\begin{aligned} Y_2(z) &= \frac{1}{M}\sum_{k=0}^{M-1} X_2(W_M^k z^{1/M}) \\ X_2(z) &= X(z^L) \end{aligned}\right\} \Rightarrow Y_2(z) = \frac{1}{M}\sum_{k=0}^{M-1} X(W_M^{kL} z^{L/M})$$

need to raise the whole term to Lth power

$W_M^k = e^{-j\frac{2\pi k}{M}}$, $k = 0, \ldots M-1$ are M distinct $M^{th}$ roots of unity

$W_M^{kL} = e^{-j\frac{2\pi kL}{M}}$, $k = 0, \ldots M-1$ may not represent M distinct numbers when L and M share common factors.

# Proof of Noble Identities

Proof:

(a) $$Y_2(z) = \frac{1}{M}\sum_{k=0}^{M-1} X_2(W_M^k z^{1/M})$$

$$X_2(z) = G(z^M)X(z) \Rightarrow X_2(W_M^k z^{1/M}) = G(W_M^{Mk} z)X(W_M^k z^{1/M})$$

$$\therefore Y_2(z) = \frac{1}{M}\sum_{k=0}^{M-1} G(z)X(W_M^k z^{1/M})$$

$$= G(z)X_1(z) = Y_1(z)$$

(b) $$Y_4(z) = G(z^L)X_4(z) = G(z^L)X(z^L)$$

$$\left.\begin{array}{l} Y_3(z) = X_3(z^L) \\ X_3(z) = G(z)X(z) \end{array}\right\} \Rightarrow Y_3(z) = G(z^L)X(z^L) = Y_4(z) \quad \blacksquare$$