1. Assume that $v(n)$ is a real-valued zero-mean white Gaussian noise with $\sigma_v^2 = 1$, $x(n)$ and $y(n)$ are generated by the equations

$$x(n) = 0.5x(n-1) + v(n),$$

$$y(n) = x(n-1) + x(n).$$

(a) Find the power spectrum of sequence $x(n)$, and its power.

(b) Find the power spectrum of sequence $y(n)$, and its power.

(c) Calculate $r_y(k)$ for $k = 0, 1, 2, 3$.

Assume now we don't know the real model of the signal, and we want to estimate its power spectrum from $r_y(k)$ obtained in part (c). Estimate power spectrum using the following methods:

(d) ARMA(1,1) spectral estimation.

(e) AR(2) spectral estimation.

(f) Maximum entropy spectral estimation with order 2.

(g) Minimum variance spectral estimation with order 1.

2. Show that the periodogram spectrum estimator will result in biased results if an N-point rectangular window is applied, i.e., $P_{PER}(\omega) = \frac{1}{N}|\sum_{n=0}^{N-1} x(n)e^{-j\omega n}|^2$ is biased.

Solution:

$E[P_{PER}(\omega)] = E[\frac{1}{N}|\sum_{n=0}^{N-1} x(n)e^{-j\omega n}|^2] = \frac{1}{N}E[\sum_{n=0}^{N-1} x(n)e^{-j\omega n}\sum_{m=0}^{N-1} x^*(m)e^{j\omega m}]$ $= \frac{1}{N}\sum_{n=0}^{N-1}\sum_{m=0}^{N-1} E[x(n)x^*(m)]e^{-j\omega(n-m)} = \frac{1}{N}\sum_{n=0}^{N-1}\sum_{m=0}^{N-1} r(n-m)e^{-j\omega(n-m)} = \frac{1}{N}\sum_{l=-(N-1)}^{N-1}(N-|l|)r(l)e^{-j\omega l} = \sum_{l=-(N-1)}^{N-1}(1-|l|/N)r(l)e^{-j\omega l}$

Note that the true spectrum is the Fourier transform of $\{r(l)\}$, i.e., $P(w) = \sum_l r(l)e^{-j\omega l}$. As a result, $E[P_{PER}(\omega)] = P(w) * w_T(\omega)$ is a "smeared" version of the true spectrum, where $w_T(\omega)$ is the Fourier transform of a triangle waveform (and hence has the form of $\text{sinc}(\cdot)^2$).

3. Consider a wide-sense stationary process consisting of p distinct complex sinusoids in white noise with variance σ^2, i.e.

$$x(n) = \left[\sum_{i=1}^{p} A_i e^{-j(n\omega_i+\phi_i)}\right] + v(n)$$

where A_i and ϕ_i are uncorrelated, and ϕ_i is a uniformly distributed random variable in $[0, 2\pi)$.

(a) Find the autocorrelation function $r(k) = E[x(n)x(n-k)]$.

(b) Find the $(p+1)\times(p+1)$ correlation matrix R.

4. Consider a random process

$$x(n) = A\exp[j(n\omega_0 + \phi)] + \alpha_0 v[n] + \alpha_1 v[n-1],$$

where $\{v[n]\}$ is a white noise process with zero mean and variance σ_v^2. The phase ϕ is uniformly distributed over $[0, 2\pi)$ and uncorrelated with $v[n]$; and A, ω_0, α_0, and α_1 are real-valued constants.

(a) Find the autocorrelation function for $\{x[n]\}$ in terms of $A, \omega_0, \alpha_0, \alpha_1$, and σ_v^2. Your solution should provide all the necessary steps and justifications.

(b) Consider the process in $\{x[n]\}$ for the case of $\alpha_0 = 1$ and $\alpha_1 = 0$. First, determine the eigen values of an $M \times M$ correlation matrix of the $\{x[n]\}$ process. Next. suppose we have observed N samples, $x[0], x[1], ..., x[N-1]$. Use equation, diagram, and concise words to describe the average periodogram method for estimating method for estimating the power spectrum density of the $\{x[n]\}$ process.

5. Assume the signal $x(n) = acos(\omega n + \phi) + v(n)$, where a is an unknown constant, $v(n)$ is a white Gaussian noise independent of the sinusoid. Suppose we know the autocorrelation coefficients $r(0) = 3$, $r(1) = \sqrt{2}$, and $r(2) = 0$, determine the frequency of the sinusoid ω and the noise power σ_v^2.

Solution:

The cosine wave is two exponential signals with frequencies $\pm\omega$. We have to use 3×3 correlation matrix,

$$\boldsymbol{R} = \begin{bmatrix} 3 & \sqrt{2} & 0 \\ \sqrt{2} & 3 & \sqrt{2} \\ 0 & \sqrt{2} & 3 \end{bmatrix}.$$

The eigenvalues are $1, 3, 5$; the eigenvector corresponding to the minimum eigenvalue is $(1, -\sqrt{2}, 1)^T$. According to the MUSIC/Pisorenko algorithm, $\sigma_v^2 = 1$, $1 - \sqrt{2}e^{j\omega} + e^{j2\omega} = 0$. Solving the equation, we get $\omega = \pi/4$.