1. Determine if each of the following are valid autocorrelation matrices of WSS processes. (Correlation Matrix)

$$\boldsymbol{R}_a = \begin{bmatrix} 4 & 1 & 1 \\ -1 & 4 & 1 \\ -1 & -1 & 4 \end{bmatrix}, \boldsymbol{R}_b = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}, \boldsymbol{R}_c = \begin{bmatrix} 2j & 0 & j \\ 0 & 2j & 0 \\ -j & 0 & 2j \end{bmatrix}, \boldsymbol{R}_d = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}.$$

Solution:

Recall that the properties of an autocorrelation matrix for a WSS process is that (1) $\boldsymbol{R}$ is Toeplitz; (2) $\boldsymbol{R}^H = \boldsymbol{R}$; (3) $\boldsymbol{R}$ is non-negative definite.

$\boldsymbol{R}_a$ is NOT Hermitian; $\boldsymbol{R}_b$ is NOT Toeplitz; $\boldsymbol{R}_c$ is NOT Hermitian; $\boldsymbol{R}_d$ is NOT non-negative definite ($\lambda = 1, -1, 3$).

2. Consider the random process $y(n) = x(n) + v(n)$, where $x(n) = Ae^{j(\omega n+\phi)}$ and $v(n)$ is zero mean white Gaussian noise with a variance σ_v^2. We also assume the noise and the complex sinusoid are independent. Under the following conditions, determine if $y(n)$ is WSS. Justify your answers. (WSS Process)

(a) ω and A are constants, and ϕ is a uniformly distributed over the interval $[0, 2\pi]$.

(b) ω and ϕ are constants, and A is a Gaussian random variable $\sim \mathcal{N}(0, \sigma_A^2)$.

(c) ϕ and A are constants, and ω is a uniformly distributed over the interval $[\omega_0 - \Delta, \omega_0 + \Delta]$ for some fixed Δ.

Solution:

(a)

$$\begin{aligned} E[y(n)] &= Ae^{j\omega n}E_\phi[e^{j\phi}] + E_v[v(n)] = 0 \\ E[y(n)y^*(n-k)] &= E_\phi[(Ae^{j(\omega n+\phi)} + v(n))(A^*e^{-j(\omega(n-k)+\phi)} + v^*(n-k))] \\ &= |A|^2 E_\phi[e^{j\omega k}] + \sigma_v^2\delta(k) \\ &= |A|^2 e^{j\omega k} + \sigma_v^2\delta(k) \end{aligned}$$

1st and 2nd moments are independent of n. Thus, the process is WSS.

(b)

$$\begin{aligned} E[y(n)] &= E_A[A]e^{j(\omega n+\phi)} + E_v[v(n)] = 0 \\ E[y(n)y^*(n-k)] &= E_A[(Ae^{j(\omega n+\phi)} + v(n))(A^*e^{-j(\omega(n-k)+\phi)} + v^*(n-k))] \\ &= E_A[AA^*]e^{j\omega k} + \sigma_v^2\delta(k) \\ &= \sigma_A^2 e^{j\omega k} + \sigma_v^2\delta(k) \end{aligned}$$

1st and 2nd moments are independent of n. Thus, the process is WSS.

(c)

$$\begin{aligned}
E[y(n)] &= E_\omega[x(n)] + E_v[v(n)] = A \cdot E_\omega[e^{j\omega n}] \cdot e^{j\phi} = \frac{Ae^{j\phi}}{2jn\Delta} e^{j\omega n}\Big|_{\omega_0-\Delta}^{\omega_0+\Delta} \\
\Rightarrow |E[y(n)]| &\leq |\frac{Ae^{j\phi}}{2jn\Delta}| \cdot 2 \to 0 \text{ as } n \to \infty \\
E[y(n)y^*(n-k)] &= E_\omega[(Ae^{j(\omega n+\phi)} + v(n))(A^* e^{-j(\omega(n-k)+\phi)} + v^*(n-k))] \\
&= |A|^2 E_\omega[e^{j\omega k}] + \sigma_v^2 \delta(k) \\
&= |A|^2 e^{j\omega_0 k} \frac{\sin(k\Delta)}{k\Delta} + \sigma_v^2 \delta(k)
\end{aligned}$$

The sequence defined here is actually NOT a WSS process, but its 1st and 2nd moment statistics are approximately independent of n as $n \to \infty$.

3. [Rec.II P2(a) revisited] Determine the PSD of the WSS process $y(n) = Ae^{j(\omega_0 n+\phi)} + v(n)$, where $v(n)$ is zero mean white Gaussian noise with a variance σ_v^2, and ϕ is uniformly distributed over the interval $[0, 2\pi]$. (Power Spectral Density)

Solution:

In the autocorrelation function in P2(a) is

$$r_y(k) = A^2 e^{j\omega k} + \sigma_v^2 \delta(k)$$

By taking discrete time Fourier transform on $r_y(k)$, we get

$$P_y(\omega) = 2\pi A^2 \delta(\omega - \omega_0) + \sigma_v^2$$

4. Assume $v(n)$ is a white Gaussian random process with zero mean and variance 1. The two filters in Fig. RII.4 are $G(z) = \frac{1}{1-0.4z^{-1}}$ and $H(z) = \frac{2}{1-0.5z^{-1}}$. (Auto-Regressive Process)

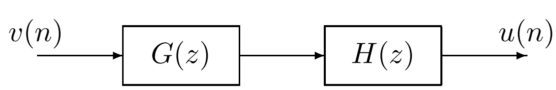

Figure RII.4:

(a) Is $u(n)$ an AR process? If so, find the parameters.

(b) Find the autocorrelation coefficients $r_u(0)$, $r_u(1)$, and $r_u(2)$ of the process $u(n)$.

Solution:

(a) $U(z) = \frac{2}{1-0.9z^{-1}+0.2z^{-2}} V(z)$, $u(n) = 0.9u(n-1) - 0.2u(n-2) + 2v(n)$, $a_1 = -0.9$, $a_2 = 0.2$.

(b) Apply the Yule-Walker equation,

$$\begin{pmatrix} r_u(0) & r_u(1) \\ r_u(1) & r_u(0) \end{pmatrix} \begin{pmatrix} -0.9 \\ 0.2 \end{pmatrix} = -\begin{pmatrix} r_u(1) \\ r_u(2) \end{pmatrix},$$

from which we get

$$\begin{cases} r_u(1) = -\frac{a_1}{1+a_2} r_u(0) = \frac{3}{4} r_u(0) \\ r_u(2) = \left(\frac{a_1^2}{1+a_2} - a_2\right) r_u(0) = \frac{19}{40} r_u(0) \end{cases}$$

Moreover, since $r_u(0) + a_1 r_u(1) + a_2 r_u(2) = 4\sigma_v^2$ (Here, '4' because in this model it is '$2v(n)$' rather than '$v(n)$'), we have $r_u(0) = \frac{1+a_2}{1-a_2} \frac{4\sigma_v^2}{(1+a_2)^2 - a_1^2} = \frac{200}{21}$. Then, $r_u(1) = \frac{50}{7}$, and $r_u(2) = \frac{95}{21}$.

Note:

1. In general, for a p-order AR model, given $\{\sigma_v^2, a_1, a_2, \ldots, a_p\}$, we can find $\{r(0), r(1), r(2), \ldots\}$; and vice versa. They are related by Yule-Walker Equations.

2. $r(-k) = r^*(k)$ in general (and hence matrix $\boldsymbol{R}$ is Hermitian), and $r(-k) = r(k)$ for real-valued signals. $r(0)$ is the power of sequence $u(n)$, and hence $r(0) > 0$ from physical point of view.

3. For an AR model, $u(n) = \sum_{k=1}^{p} -a_k u(n-k) + v(n)$ has NO correlation with future $v(m), m = n+1, n+2, \ldots$ (convince yourself). Simply multiply both sides by $u^*(n)$ and take expectation, we get $r(0) = \sum_{k=1}^{p} -a_k r(-k) + E(v(n)u^*(n))$. Note that $E(v(n)u^*(n)) = E(v(n)(\sum_{k=1}^{p} -a_k^* u^*(n-k) + v^*(n)))$ but $E(v(n)u^*(n-k)) = 0$ for $k \geq 1$. Then, $r(0) = \sum_{k=1}^{p} -a_k r(-k) + \sigma_v^2$, which we have used to find the relation of $r(0)$ (signal power) and σ_v^2 (model parameter) in part (b). We could multiply $u^*(n-k)$ instead of $u^*(n)$ and take the expectation, and this is how the Yule-Walker equations are derived.

5. Let a real-valued AR(2) process be described by

$$u(n) = x(n) + a_1 x(n-1) + a_2 x(n-2)$$

where $u(n)$ is a white noise of zero-mean and variance σ^2, and $u(n)$ and past values $x(n-1)$, $x(n-2)$ are uncorrelated. (Yule-Walker Equation)

(a) Determine and solve the Yule-Walker Equations for the AR process.

(b) Find the variance of the process $x(n)$.

Solution: (a) Solve the Yule-Walker equation, we have

$$\begin{aligned} r_x(0) &= -a_1 r_x(-1) - a_2 r_x(-2) + \sigma^2 \\ r_x(1) &= -a_1 r_x(0) - a_2 r_x(-1) \\ r_x(2) &= -a_1 r_x(1) - a_2 r_x(0) \end{aligned}$$

Use the relation that $r_x(k) = r_x(-k)$ and solve this we get

$$
\begin{aligned}
r_x(0) &= \frac{\sigma^2}{1 - \frac{a_1^2}{1+a_2} + a_2(\frac{a_1^2}{1+a_2} - a_2)} \\
r_x(1) &= -\frac{a_1}{1+a_2} r_x(0) \\
r_x(2) &= (\frac{a_1^2}{1+a_2} - a_2) r_x(0)
\end{aligned}
$$

(b) The process is zero mean, so the variance is $r_x(0)$.

6. [Problem II.4 continued] Assume $v(n)$ and $w(n)$ are white Gaussian random processes with zero mean and variance 1. The two filters in Fig. RII.6 are $G(z) = \frac{1}{1-0.4z^{-1}}$ and $H(z) = \frac{2}{1-0.5z^{-1}}$. (Wiener Filter)

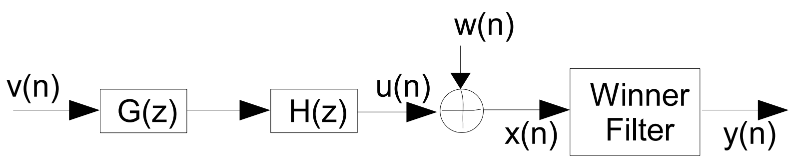

Figure RII.6:

(a) Design a 1-order Wiener filter such that the desired output is $u(n)$. What is the MSE?

(b) Design a 2-order Wiener filter. What is the MSE?

Solution:

(a) $\boldsymbol{R}_x = \begin{bmatrix} r_u(0)+1 & r_u(1) \\ r_u(1) & r_u(0)+1 \end{bmatrix}$, and $\boldsymbol{p}_{xd} = \begin{bmatrix} r_u(0) \\ r_u(1) \end{bmatrix}$. The filter is $\boldsymbol{w} = \boldsymbol{R}_x^{-1}\boldsymbol{p}$ with MSE $r_u(0) - \boldsymbol{p}_{xd}^H \boldsymbol{R}_x^{-1} \boldsymbol{p}_{xd}$.

(b) Similar to (a), except $\boldsymbol{R}_x = \begin{bmatrix} r_u(0)+1 & r_u(1) & r_u(2) \\ r_u(1) & r_u(0)+1 & r_u(1) \\ r_u(2) & r_u(1) & r_u(0)+1 \end{bmatrix}$, and $\boldsymbol{p}_{xd} = \begin{bmatrix} r_u(0) \\ r_u(1) \\ r_u(2) \end{bmatrix}$. MSE is still the same expression, i.e. $r_u(0) - \boldsymbol{p}_{xd}^H \boldsymbol{R}_x^{-1} \boldsymbol{p}_{xd}$.

Note:

1. In general, for a p-order AR model, given $\{\sigma_v^2, a_1, a_2, \ldots, a_p\}$, we can find $\{r(0), r(1), r(2), \ldots\}$; and vice versa. They are related by Yule-Walker Equations.

2. $r(-k) = r^*(k)$ in general (and hence matrix $\boldsymbol{R}$ is Hermitian), and $r(-k) = r(k)$ for real-valued signals. $r(0)$ is the power of sequence $u(n)$, and hence $r(0) > 0$ from physical point of view.

3. For an AR model, $u(n) = \sum_{k=1}^{p} -a_k u(n-k) + v(n)$ has NO correlation with future $v(m), m = n+1, n+2, \ldots$ (convince yourself). Simply multiply both sides by $u^*(n)$ and take expectation, we get $r(0) = \sum_{k=1}^{p} -a_k r(-k) + E(v(n)u^*(n))$. Note that $E(v(n)u^*(n)) = E(v(n)(\sum_{k=1}^{p} -a_k^* u^*(n-k) + v^*(n)))$ but $E(v(n)u^*(n-k)) = 0$ for $k \geq 1$. Then, $r(0) = \sum_{k=1}^{p} -a_k r(-k) + \sigma_v^2$, which we have used to find the relation of $r(0)$ (signal power) and σ_v^2 (model parameter) in part (b). We

could multiply $u^*(n-k)$ instead of $u^*(n)$ and take the expectation, and this is how the Yule-Walker equations are derived.

4. When designing Wiener filtering, one should find $\boldsymbol{R}_{xx}$ and $\boldsymbol{p}_{xd}$ first. Then, it's straightforward to apply $\boldsymbol{w} = \boldsymbol{R}_{xx}^{-1}\boldsymbol{p}_{xd}$ with MSE $\sigma_d^2 - \boldsymbol{p}_{xd}^H \boldsymbol{R}_{xx}^{-1}\boldsymbol{p}_{xd}$.

7. The autocorrelation sequence of a given zero-mean real-valued random process $u(n)$ is $r(0) = 1.25$, $r(1) = r(-1) = 0.5$, and $r(k) = 0$ for any $|k| \geq 2$. (Wiener Filter)

(a) What model fits this process best: AR or MA? Find the corresponding parameters.

(b) Design the Wiener filter when using $u(n)$ to predict $u(n+1)$. Can we do better (in terms of MSE) if we use both $u(n)$ and $u(n-1)$ as the input to the Wiener filter? What if using $u(n)$ and $u(n-2)$?

Solution:

(a) Apparently, it is an MA process with order 1, i.e., $x(n) = v(n) + bv(n-1)$, $v(n)$ is a zero-mean white sequence with variance σ_v^2.

Then, $r(0) = E(x(n)x^*(n)) = (1+|b|^2)\sigma_v^2$, and $r(1) = E(x(n)x^*(n-1)) = b\sigma_v^2$. We can find two solutions $(b = 2, \sigma_v^2 = 0.25)$ and $(b = 0.5, \sigma_v^2 = 1)$.

(b1) $R = E(u(n)u^*(n)) = r(0)$, and $p = E(u(n)u^*(n+1)) = r(-1)$. Hence, $w = r(0)^{-1}r(-1) = 2/5$, i.e., $y(n) = 2/5u(n)$ and MSE $= 1.25 - 0.2 = 1.05$.

(b2) $\boldsymbol{R} = E(\binom{u(n)}{u(n-1)}[u^*(n), u^*(n-1)]) = \begin{pmatrix} r(0) & r(1) \\ r(1) & r(0) \end{pmatrix} = \begin{pmatrix} 1.25 & 0.5 \\ 0.5 & 1.25 \end{pmatrix}$, and $\boldsymbol{p} = E(\binom{u(n)}{u(n-1)}u^*(n+1)) = \binom{0.5}{0}$. $y(n) = 10/21u(n) - 4/21u(n-1)$, and MSE $= 1.25 - 5/21 \simeq 1.01$. Improved.

(b3) $\boldsymbol{R} = E(\binom{u(n)}{u(n-2)}[u^*(n), u^*(n-2)]) = \begin{pmatrix} r(0) & r(2) \\ r(2) & r(0) \end{pmatrix} = \begin{pmatrix} 1.25 & 0 \\ 0 & 1.25 \end{pmatrix}$, and $\boldsymbol{p} = E(\binom{u(n)}{u(n-2)}u^*(n+1)) = \binom{0.5}{0}$. $y(n) = 2/5u(n) + 0u(n-2)$ which is exactly the same with (b1).

8. Consider the MIMO (multi-input multi-output) wireless communications system shown in Fig. RII.8. There are two antennas at the transmitter and three antennas at the receiver. Assume the channel gain from the i-th transmit antenna to the j-th receive antenna is h_{ji}. Take a snapshot at time slot n, the received signal is $y_j(n) = h_{j1}x_1(n) + h_{j2}x_2(n) + v_j(n)$ where $v_j(n)$ are white Gaussian noise (zero mean, variance N_0) independent of signals. We further assume $x_1(n)$ and $x_2(n)$ are uncorrelated, and their power are P_1 and P_2, respectively. Use $y_1(n), y_2(n)$ and $y_3(n)$ as input, find the optimal Wiener filter to estimate $x_1(n)$ and $x_2(n)$. (Wiener Filter)

Solution:

Denote $\boldsymbol{y}(n) = [y_1(n), y_2(n), y_3(n)]^T$, and $\boldsymbol{v}(n) = [v_1(n), v_2(n), v_3(n)]^T$. We can have a matrix representation of the system: $\boldsymbol{y}(n) = \boldsymbol{H}\boldsymbol{x}(n) + \boldsymbol{v}(n)$.

For Wiener filters, we need to find the autocorrelation matrix of the input to the filter, and the cross-correlation vector of the input and the desired output. (It's not a big deal whether such signals are in time domain or other domain, e.g., space domain.)

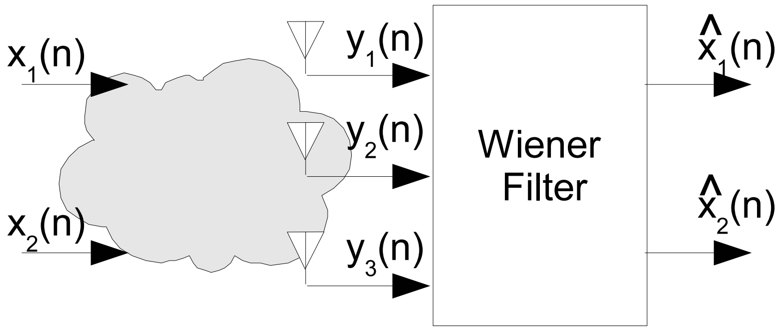

Figure RII.8:

$\boldsymbol{R}_{yy} = E[\boldsymbol{y}(n)\boldsymbol{y}(n)^H] = E[(\boldsymbol{Hx}(n)+\boldsymbol{v}(n))(\boldsymbol{Hx}(n)+\boldsymbol{v}(n))^H] = E[\boldsymbol{Hx}(n)\boldsymbol{x}^H(n)\boldsymbol{H}^H]+E[\boldsymbol{v}(n)\boldsymbol{v}(n))^H] = \boldsymbol{H}\begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix}\boldsymbol{H}^H + N_0\boldsymbol{I}$.

$$\boldsymbol{r}_{yx1} = E[\boldsymbol{y}(n)x_1(n)^*] = E[\boldsymbol{Hx}(n)x_1(n)^*] = P_1 \begin{bmatrix} h_{11} \\ h_{21} \\ h_{31} \end{bmatrix}.$$

Then, $\boldsymbol{w}_1 = \boldsymbol{R}_{yy}^{-1}\boldsymbol{r}_{yx1}$. The output is $\hat{x}_1(n) = P_1[h_{11}^*, h_{21}^*, h_{31}^*]\boldsymbol{R}_{yy}^{-1}\boldsymbol{y}(n)$.

Similar for $\boldsymbol{w}_2$.

9. Given an real-valued AR(3) model with parameters $\Gamma_1 = -4/5$, $\Gamma_2 = 1/9$, $\Gamma_3 = 1/8$, and $r(0) = 1$. Find $r(1), r(2)$, and $r(3)$. (Levinson-Durbin Recursion)

10. Consider the MA(1) process $x(n) = v(n) + bv(n-1)$ with $v(n)$ being a zero-mean white sequence with variance 1. If we use Γ_k to represent this system, prove that (Levinson-Durbin Recursion)

$$\Gamma_{m+1} = \frac{\Gamma_m^2}{\Gamma_{m-1}(1-|\Gamma_m|^2)}.$$

11. Given a p-order AR random process $\{x(n)\}$, it can be equivalently represented by any of the three following sets of values: (Levinson-Durbin Recursion)

- $\{r(0), r(1), \ldots, r(p)\}$
- $\{a_1, a_2, \ldots, a_p\}$ and $r(0)$
- $\{\Gamma_1, \Gamma_2, \ldots, \Gamma_p\}$ and $r(0)$

(a) If a new random process is defined as $x'(n) = cx(n)$ where c is a real-valued constant, what will be the new autocorrelation sequence $r'(k)$ in terms of $r(k)$ (for $k = 1, 2, \ldots, p$)? How about a'_k and Γ'_k?

(b) Let a new random process be defined as $x'(n) = (-1)^n x(n)$. Prove that $r'(k) = (-1)^k r(k)$, $a'_k = (-1)^k a_k$ and $\Gamma'_k = (-1)^k \Gamma_k$. (Hint: use induction when proving Γ_k, since Γ_k is calculated recursively.)

12. Given a lattice predictor that simultaneously generate both forward and backward prediction errors $f_m(n)$ and $b_m(n)$ ($m = 1, 2, \ldots, M$). (Lattice Structure)

(a) Find $E(f_m(n)b_i^*(n))$ for both conditions when $i \leq m$ and $i > m$.

(b) Find $E(f_m(n+m)f_i^*(n+i))$ for both conditions when $i = m$ and $i < m$.

(c) Design a joint process estimation scheme using the forward prediction errors.

(d) If for some reason we can only obtain part of forward prediction error (from order 0 to order k) and part of backward prediction error (from oder $k+1$ to order M), i.e., we have $\{f_0(n), f_1(n), \ldots, f_k(n), b_{k+1}(n), b_{k+2}(n), \ldots, b_M(n)\}$. Describe how to use such mixed forward and backward prediction errors to perform joint process estimation.

(Hint: the results from (a) and (b) will be useful for questions (c) and (d).)

13. Consider the backward prediction error sequence $b_0(n), b_1(n), \ldots, b_M(n)$ for the observed sequence $\{u(n)\}$. (Properties of FLP and BLP Errors)

(a) Define $\boldsymbol{b}(n) = [b_0(n), b_1(n), \ldots, b_M(n)]^T$, and $\boldsymbol{u}(n) = [u(n), u(n-1), \ldots, u(n-M)]^T$, find $\boldsymbol{L}$ in terms of the coefficients of the backward prediction-error filter where $\boldsymbol{b}(n) = \boldsymbol{L}\boldsymbol{u}(n)$.

(b) Let the correlation matrix for $\boldsymbol{b}(n)$ be $\boldsymbol{D}$, and that for $\boldsymbol{u}(n)$ be $\boldsymbol{R}$. Is $\boldsymbol{D}$ diagonal? What is relation between $\boldsymbol{R}$ and $\boldsymbol{D}$? Show that a lower triangular matrix $\boldsymbol{A}$ exists such that $\boldsymbol{R}^{-1} = \boldsymbol{A}^H\boldsymbol{A}$.

(c) Now we are to perform joint estimation of a desired sequence $\{d(n)\}$ by using either $\{b_k(n)\}$ or $\{u(n)\}$, and their corresponding optimal weight vectors are $\boldsymbol{k}$ and $\boldsymbol{w}$, respectively. What is relation between $\boldsymbol{k}$ and $\boldsymbol{w}$?