ENEE 630 Homework 6[1]

Material Covered: Application of Wiener Filter and the Method of Lagrange Multipliers.

Problem 1 The tap-input vector of a transversal filter is defined by

$$\mathbf{u}(n) = \alpha(n)\mathbf{s}(\omega) + \mathbf{v}(n)$$

where

$$\begin{aligned}\mathbf{s}(\omega) &= [1, e^{-j\omega}, \ldots, e^{-j\omega(M-1)}]^T \\ \mathbf{v}(n) &= [v(n), v(n-1), \ldots, v(n-M+1)]^T\end{aligned}$$

i.e. $u(n-k) = \alpha(n) \cdot e^{-jk\omega} + v(n-k)$ for $k = 0, \ldots, M-1$. For the tap-input vector at a given time n, $\alpha(n)$ is a complex random variable with zero mean and variance $\sigma_\alpha^2 = E[|\alpha(n)|^2]$, and $\alpha(n)$ is uncorrelated with the w.s.s. process $\mathbf{v}(n)$.

(a) Determine the correlation matrix of the tap-input vector $\mathbf{u}$(n).

(b) Suppose that the desired response d(n) is uncorrelated with $\mathbf{u}$(n). What is the value of the tap-weight vector of the corresponding Wiener filter?

(c) Suppose that the variance σ_α^2 is zero, and the desired response is defined by

$$d(n) = v(n-k)$$

where $0 \le k \le M-1$. What is the new value of the tap-weight vector of the Wiener filter?

(d) Determine the tap-weight vector of the Wiener filter for a desired response defined by

$$d(n) = \alpha(n)e^{-jw\tau}$$

where τ is a prescribed delay.

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Note: The mathematical tools that you may need to solve Problems 2,3, and 4 may include: (1)matrix inversion lemma, (2)differentiation with respect to complex variables or vectors, and (3)the method of Lagrange multipliers. All these are reviewed in Hayes' book, and in Haykin's book (main text or appendices depending on the edition).

Detail hint: Construct a Lagrange function that is real valued. Let $f(\mathbf{w})$ be the expression of the constraint. You may construct a real valued version of the constraint expression by $Re\{2f(\mathbf{w})\} = f(\mathbf{w}) + f^*(\mathbf{w})$. Recall we discussed in lecture that when taking partial derivative, consider $\mathbf{w}$ and $\mathbf{w}^*$ as independent parameters.

Problem 2 In this problem we explore an application of Wiener filtering to *radar*. The sampled form of the transmitted radar signal is $A_0 e^{jw_0 n}$ where w_0 is the transmitted angular frequency, and A_0 is the transmitted complex amplitude. The received signal is

$$u(n) = A_1 e^{jw_1 n} + v(n)$$

where $|A_1| < |A_0|$ and w_1 differs from w_0 by virtue of the *Doppler* shift produced by the motion of a target of interest, and v(n) is a sample of white noise that is uncorrelated with A_1.

(a) Show that the correlation matrix of the time series {u(n)}, made up of M elements, may be written as

$$\mathbf{R} = \sigma_v^2 \mathbf{I} + \sigma_1^2 \mathbf{s}(w_1)\mathbf{s}^H(w_1)$$

where σ_v^2 is the variance of the zero-mean white white noise v(n), and

$$\begin{aligned} \sigma_1^2 &= E[|A_1|^2] \\ \mathbf{s}(w_1) &= [1, e^{-jw_1}, \ldots, e^{-jw_1(M-1)}]^T \end{aligned}$$

And what is R^{-1}?

(b) The time series {u(n)} is applied to an M-tap Wiener filter with the cross-correlation vector $\mathbf{p}$ between {u(n)} and the desired response d(n) preset to

$$\mathbf{p} = \sigma_0^2 \mathbf{s}(w_0)$$

where

$$\sigma_0^2 = E[|A_0|^2]$$
$$\mathbf{s}(w_0) = [1, e^{-jw_0}, \ldots, e^{-jw_0(M-1)}]^T$$

Derive an expression for the tap-weight vector of the Wiener filter.

Problem 3 A linear array consists of M uniformly spaced sensors. The individual sensor outputs are weighted and then summed, producing the output

$$e(n) = \sum_{k=1}^{M} w_k^* u_k(n)$$

where $u_k(n)$ is the output of sensor k at time n, and w_k is the associated weight. The weights are chosen to minimize the mean-square value of e(n), subject to the constraint

$$\mathbf{w}^H \mathbf{s} = 1$$

where **s** is a prescribed steering vector. By using the method of Lagrange multipliers, show that the optimum value of the vector **w** is

$$\mathbf{w}_0 = \frac{\mathbf{R}^{-1}\mathbf{s}}{\mathbf{s}^H \mathbf{R}^{-1} \mathbf{s}}$$

where **R** is the spatial correlation matrix of the linear array.

Problem 4 Consider a discrete-time stochastic process {u(n)} that consists of K (uncorrelated) complex sinusoids plus additive white noise of zero mean and variance σ^2. That is,

$$u(n) = \sum_{k=1}^{K} A_k e^{jw_k n} + v(n)$$

where the terms $A_k e^{jw_k n}$ and v(n) refer to the k^{th} sinusoid and noise, respectively. The process {u(n)} is applied to a transversal filter with M taps, producing the output

$$e(n) = \mathbf{w}^H \mathbf{u}(n)$$

Assume that $M > K$. The requirement is to choose the tap-weight vector **w** so as to minimize the mean-square value of e(n), subject to the multiple signal-protection constraint

$$\mathbf{S}^H \mathbf{w} = \mathbf{D}^{\frac{1}{2}} \mathbf{1}$$

where $\mathbf{S}$ is the $M \times K$ signal matrix whose k^{th} column has 1, $e^{jw_k}, \ldots, e^{jw_k(M-1)}$ for its elements, $\mathbf{D}$ is the $K \times K$ diagonal matrix whose nonzero elements equal the average powers of the individual sinusoids, and the $K \times 1$ vector $\mathbf{1}$ has 1's for all its elements. Using the method of Lagrange multipliers, show that the value of the optimum weight vector that results from the constraint optimization equals

$$\mathbf{w}_0 = \mathbf{R}^{-1}\mathbf{S}(\mathbf{S}^H\mathbf{R}^{-1}\mathbf{S})^{-1}\mathbf{D}^{\frac{1}{2}}\mathbf{1}$$

where $\mathbf{R}$ is the correlation matrix of the $M \times 1$ tap-input vector $\mathbf{u}$(n).

[Note: The vector $\mathbf{w_0}$ in Problem-3 is known as the spatial version of the Minimum Variance Distortionless Response (MVDR) from the array signal processing literature. And the result of $\mathbf{w_0}$ in Problem-4 represents a temporal generalization of the MVDR formula. For more details on MVDR beamforming method, refer to Haykin's book *Adaptive Filter Theory.*]