ENEE 621
SPRING 2017
ESTIMATION AND DETECTION THEORY

ANSWER KEY TO TEST # 1:

1.

1.a. Fix $\eta > 0$[1] and recall that

$$d_\eta(y) \quad \text{iff} \quad f_1(y) < \eta f_0(y), \quad y \in \mathbb{R}.$$

Here, with the conventions implied by the definition of d_η, we have

$y \le -1$:	$d_\eta(y) = 1$
$-1 < y \le 0$:	$d_\eta(y) = 0$
$0 < y \le 1$:	$d_\eta(y) = 0$ iff $1 < 3\eta(1-y)$
$1 < y \le 3$:	$d_\eta(y) = 1$
$3 < y$:	$d_\eta(y) = 1$

Collecting these facts we conclude that

$$\begin{aligned} C(d_\eta) &\equiv \{y \in \mathbb{R} : \ d_\eta(y) = 0\} \\ &= (-1,0) \cup \{y \in [0,1) : \ 1 < 3\eta(1-y)\}. \end{aligned} \tag{1.1}$$

1.b. For $\eta = 0$, we know that $P_{\rm F}(d_\eta) = 1$ and $P_{\rm D}(d_\eta) = 1$ as explained in the Lecture Notes. From now on, fix $\eta > 0$. We shall write

$$t(\eta) \equiv \left(1 - \frac{1}{3\eta}\right)^+ = \begin{cases} 0 & \text{if } 0 < \eta \le \frac{1}{3} \\ 1 - \frac{1}{3\eta} & \text{if } \frac{1}{3} \le \eta. \end{cases}$$

[1] For $\eta = 0$, the test d_0 always selects the alternative.

Obviously,

$$\begin{aligned}
P_{\mathrm{F}}(d_\eta) &= \mathbb{P}\left[d_\eta(Y) = 1 | H = 0\right] \\
&= 1 - \mathbb{P}\left[d_\eta(Y) = 0 | H = 0\right] \\
&= 1 - \mathbb{P}\left[-1 < Y \le 0 | H = 0\right] - \mathbb{P}\left[0 < Y \le 1, 1 < 3\eta(1-Y) | H = 0\right] \\
&= 1 - \frac{1}{2} - \mathbb{P}\left[0 < Y \le 1, 1 < 3\eta(1-Y) | H = 0\right] \\
&= \frac{1}{2} - \mathbb{P}\left[0 < Y \le 1, 0 < Y < 1 - \frac{1}{3\eta} \middle| H = 0\right] \\
&= \frac{1}{2} - \int_0^{\left(1-\frac{1}{3\eta}\right)^+} f_0(y)dy \\
&= \frac{1}{2} - \int_0^{t(\eta)} (1-y)dy \\
&= \frac{1}{2} - \left[y - \frac{y^2}{2}\right]_0^{t(\eta)} \\
&= \frac{1}{2}\left(1 - t(\eta)\right)^2 \\
&= \begin{cases} \frac{1}{2} & \text{if } 0 < \eta \le \frac{1}{3} \\ \\ \frac{1}{18\eta^2} & \text{if } \frac{1}{3} \le \eta. \end{cases} \qquad (1.2)
\end{aligned}$$

In a similar way, we get

$$\begin{aligned}
P_{\mathrm{D}}(d_\eta) &= \mathbb{P}\left[d_\eta(Y) = 1 | H = 1\right] \\
&= 1 - \mathbb{P}\left[d_\eta(Y) = 0 | H = 1\right] \\
&= 1 - \mathbb{P}\left[-1 < Y \le 0 | H = 1\right] - \mathbb{P}\left[0 < Y \le 1, 1 < 3\eta(1-Y) | H = 1\right] \\
&= 1 - \mathbb{P}\left[0 < Y \le 1, 1 < 3\eta(1-Y) | H = 1\right] \\
&= 1 - \mathbb{P}\left[0 < Y \le 1, 0 < Y < 1 - \frac{1}{3\eta} \middle| H = 1\right] \\
&= 1 - \int_0^{\left(1-\frac{1}{3\eta}\right)^+} f_1(y)dy \\
&= 1 - \frac{1}{3}\int_0^{t(\eta)} dy \\
&= 1 - \frac{1}{3}t(\eta) \\
&= \begin{cases} 1 & \text{if } 0 < \eta \le \frac{1}{3} \\ \\ \frac{2}{3} + \frac{1}{9\eta} & \text{if } \frac{1}{3} \le \eta. \end{cases} \qquad (1.3)
\end{aligned}$$

In summary we conclude that

$$P_{\mathrm{F}}(d_\eta) = \frac{1}{2}\left(1 - t(\eta)\right)^2 \quad \text{and} \quad P_{\mathrm{D}}(d_\eta) = 1 - \frac{t(\eta)}{3}, \quad \eta > 0.$$

From these expressions it is then plain that

$$\lim_{\eta\to 0} P_{\rm F}(d_\eta) = \frac{1}{2} \quad \text{and} \quad \lim_{\eta\to 0} P_{\rm D}(d_\eta) = 1$$

while

$$\lim_{\eta\to \infty} P_{\rm F}(d_\eta) = 0 \quad \text{and} \quad \lim_{\eta\to \infty} P_{\rm D}(d_\eta) = \frac{2}{3}.$$

Moreover, we get $\{P_{\rm F}(d_\eta),\ \eta > 0\} = \left(0, \frac{1}{2}\right]$ and $\{P_{\rm D}(d_\eta),\ \eta > 0\} = \left(\frac{2}{3}, 1\right]$.

1.c. With the notation introduced in the Lecture Notes we have

$$V(p) = J_p(d^\star(p)) = J_p(d_{\eta(p)}), \quad p \in (0,1]$$

where

$$\eta(p) = \frac{\Gamma_0(1-p)}{\Gamma_1 p} = \frac{1-p}{p}$$

since here $\Gamma_0 = \Gamma_1 = 1$. It is now straightforward to see that

$$\begin{aligned} V(p) &= p\mathbb{P}\left[d_{\eta(p)}(Y) = 0 | H = 1\right] + (1-p)\mathbb{P}\left[d_{\eta(p)}(Y) = 1 | H = 0\right] \\ &= p\left(1 - P_D(d_{\eta(p)})\right) + (1-p)P_F(d_{\eta(p)}) \\ &= p\left(\frac{1}{3}\left(1 - \frac{1}{3\eta(p)}\right)^+\right) + (1-p)\left(\frac{1}{2}\left(1 - \left(1 - \frac{1}{3\eta(p)}\right)^+\right)^2\right) \\ &= \frac{p}{3}\cdot\tau(p) + \frac{1-p}{2}\cdot(1-\tau(p))^2 \end{aligned}$$

with

$$\begin{aligned} \tau(p) &\equiv t(\eta(p)) \\ &= \left(1 - \frac{1}{3\eta(p)}\right)^+ \\ &= \frac{(3-4p)^+}{3(1-p)} = \begin{cases} 1 - \frac{p}{3(1-p)} & \text{if } 0 < p \le \frac{3}{4} \\ 0 & \text{if } \frac{3}{4} \le p < 1. \end{cases} \end{aligned} \tag{1.4}$$

It follows that

$$\begin{aligned} V(p) &= \begin{cases} \frac{p}{3}\left(1 - \frac{p}{3(1-p)}\right) + (1-p)\left(\frac{p^2}{18(1-p)^2}\right) & \text{if } 0 < p \le \frac{3}{4} \\ \frac{1-p}{2} & \text{if } \frac{3}{4} \le p < 1 \end{cases} \\ &= \begin{cases} \frac{p}{3}\left(1 - \frac{p}{3(1-p)}\right) + \frac{p^2}{18(1-p)} & \text{if } 0 < p \le \frac{3}{4} \\ \frac{1-p}{2} & \text{if } \frac{3}{4} \le p < 1 \end{cases} \\ &= \begin{cases} \frac{p(6-7p)}{18(1-p)} & \text{if } 0 < p \le \frac{3}{4} \\ \frac{1-p}{2} & \text{if } \frac{3}{4} \le p < 1. \end{cases} \end{aligned}$$

It is a simple matter to check that the mapping $p \to V(p)$ is concave on $[0,1]$, and differentiable on that interval except at $p = \frac{3}{r} = p_{\rm m}$.
1.d. Fix $\eta > 0$. From Part **b** we see that

$$3\left(1 - P_{\rm D}(d_\eta)\right) = t(\eta)$$

while

$$2P_{\rm F}(d_\eta) = \left(1 - t(\eta)\right)^2 = \left(1 - 3\left(1 - P_{\rm D}(d_\eta)\right)\right)^2,$$

whence

$$2P_{\rm F}(d_\eta) = \left(3P_{\rm D}(d_\eta) - 2\right)^2.$$

Therefore, since $\frac{2}{3} < P_{\rm D}(d_\eta)$ for all $\eta > 0$, it follows that

$$\sqrt{2P_{\rm F}(d_\eta)} = 3P_{\rm D}(d_\eta) - 2,$$

and we conclude that

$$P_{\rm D}(d_\eta) = \frac{2 + \sqrt{2P_{\rm F}(d_\eta)}}{3}.$$

The ROC curve is now defined through the mapping $\Gamma : [0, \frac{1}{2}] \to [\frac{2}{3}, 1]$ given by

$$P_{\rm D} = \Gamma(P_{\rm F}) = \frac{2 + \sqrt{2P_{\rm F}}}{3}, \quad 0 \le P_{\rm F} \le \frac{1}{2}.$$

Here, contrary to what happens in the "usual" case (say the Gaussian case), the ROC curve does not go from point $(0,0)$ to point $(1,1)$, but instead from $(0, \frac{2}{3})$ to point $(\frac{1}{2}, 1)$ – There is no curve defined over the entire interval $[0,1]$.

2.

For each $\theta > 0$ it is plain that $H_\theta : \ Y \sim F_\theta$ means that under H_θ the observation Y is normally distributed with zero mean and variance θ.

With distinct θ_0 and θ_1 in $(0, \infty)$, consider the binary hypothesis testing problem

$$\begin{array}{ll} H_1 : & Y \sim F_{\theta_1} \\ H_0 : & Y \sim F_{\theta_0}. \end{array} \tag{1.5}$$

For $\eta > 0$, consider the corresponding test $d_\eta : \mathbb{R} \to \{0,1\}$. In a routine manner we find

$$\begin{aligned} d_\eta(y) = 0 \quad & \text{iff} \quad f_{\theta_1}(y) < \eta f_{\theta_0}(y) \\ & \text{iff} \quad \frac{1}{\sqrt{2\pi\theta_1}} e^{-\frac{y^2}{2\theta_1}} < \eta \cdot \frac{1}{\sqrt{2\pi\theta_0}} e^{-\frac{y^2}{2\theta_0}}, \quad y \in \mathbb{R} \\ & \text{iff} \quad \left(\frac{1}{\theta_0} - \frac{1}{\theta_1}\right) y^2 < \log\left(\eta^2 \cdot \frac{\theta_1}{\theta_0}\right), \quad y \in \mathbb{R}. \end{aligned}$$

For future use, write

$$T(\eta; \theta_0, \theta_1) \equiv \left(\frac{1}{\theta_0} - \frac{1}{\theta_1}\right)^{-1} \cdot \log\left(\eta^2 \cdot \frac{\theta_1}{\theta_0}\right).$$

2.a. Assume $0 < \theta_0 < \theta_1$ – The test d_η now reads

$$d_\eta(y) = 0 \quad \text{iff} \quad y^2 < T(\eta;\theta_0,\theta_1), \quad y \in \mathbb{R}.$$

If $\eta^2 \leq \frac{\theta_0}{\theta_1}$, then $\log\left(\eta^2 \cdot \frac{\theta_1}{\theta_0}\right) \leq 0$ and $T(\eta;\theta_0,\theta_1) \leq 0$. Thus, d_η always selects the alternative H_{θ_1}, whence $\mathbb{P}_{\theta_0}[d_\eta(Y) = 1] = 1$ and $\mathbb{P}_{\theta_1}[d_\eta(Y) = 1] = 1$.

If $\frac{\theta_0}{\theta_1} < \eta^2$, then $T(\eta;\theta_0,\theta_1) > 0$ and

$$d_\eta(y) = 0 \quad \text{iff} \quad |y| < \sqrt{T(\eta;\theta_0,\theta_1)}, \quad y \in \mathbb{R}.$$

It follows that

$$\begin{aligned}
\mathbb{P}_{\theta_0}[d_\eta(Y) = 1] &= 1 - \mathbb{P}_{\theta_0}\left[-\sqrt{T(\eta;\theta_0,\theta_1)} < Y < \sqrt{T(\eta;\theta_0,\theta_1)}\right] \\
&= 1 - \mathbb{P}_{\theta_0}\left[-\sqrt{\frac{T(\eta;\theta_0,\theta_1)}{\theta_0}} < \frac{Y}{\sqrt{\theta_0}} < \sqrt{\frac{T(\eta;\theta_0,\theta_1)}{\theta_0}}\right] \\
&= 1 - \left(\Phi\left(\sqrt{\frac{T(\eta;\theta_0,\theta_1)}{\theta_0}}\right) - \Phi\left(-\sqrt{\frac{T(\eta;\theta_0,\theta_1)}{\theta_0}}\right)\right) \\
&= 2\left(1 - \Phi\left(\sqrt{\frac{T(\eta;\theta_0,\theta_1)}{\theta_0}}\right)\right). \qquad (1.6)
\end{aligned}$$

Similar calculations show that

$$\mathbb{P}_{\theta_1}[d_\eta(Y) = 1] = 2\left(1 - \Phi\left(\sqrt{\frac{T(\eta;\theta_0,\theta_1)}{\theta_1}}\right)\right). \qquad (1.7)$$

Combining the two cases we get

$$\mathbb{P}_{\theta_h}[d_\eta(Y) = 1] = 2\left(1 - \Phi\left(\sqrt{\frac{T(\eta;\theta_0,\theta_1)^+}{\theta_h}}\right)\right), \quad h = 0, 1. \qquad (1.8)$$

To obtain the Neyman-Pearson tests we proceed as follows: Fix α in $(0,1)$. We seek $\eta > 0$ such that $\mathbb{P}_{\theta_0}[d_\eta(Y) = 1] = \alpha$. This leads to the equation

$$2\left(1 - \Phi\left(\sqrt{\frac{T(\eta;\theta_0,\theta_1)^+}{\theta_0}}\right)\right) = \alpha,$$

or equivalently

$$\Phi\left(\sqrt{\frac{T(\eta;\theta_0,\theta_1)^+}{\theta_0}}\right) = 1 - \frac{\alpha}{2}.$$

Note that $1 - \frac{\alpha}{2} > \frac{1}{2}$. Any solution is characterized by

$$\frac{T(\eta;\theta_0,\theta_1)^+}{\theta_0} = \left(\Phi^{-1}\left(1-\frac{\alpha}{2}\right)\right)^2,$$

which requires $T(\eta;\theta_0,\theta_1) > 0$, namely $\frac{\theta_0}{\theta_1} < \eta^2$. Therefore, the desired η satisfies

$$\frac{T(\eta;\theta_0,\theta_1)}{\theta_0} = \left(\Phi^{-1}\left(1-\frac{\alpha}{2}\right)\right)^2 \quad \text{and} \quad \frac{\theta_0}{\theta_1} < \eta^2.$$

It is easy to check that any solution to the first equation automatically satisfies the required inequality.

Therefore, the Neyman-Pearson test $d_{\rm NP}(\alpha;\theta_0,\theta_1)$ for testing H_{θ_0} against H_{θ_1} takes the form

$$d_{\rm NP}(\alpha;\theta_0,\theta_1)(y) = 0 \quad \text{iff} \quad |y| < \sqrt{\theta_0}\cdot\Phi^{-1}\left(1-\frac{\alpha}{2}\right), \quad y \in \mathbb{R}.$$

Note that the region

$$\begin{aligned} C(d_{\rm NP}(\alpha;\theta_0,\theta_1)) &= \left\{y\in\mathbb{R} :\ |y| < \sqrt{\theta_0}\cdot\Phi^{-1}\left(1-\frac{\alpha}{2}\right)\right\} \\ &= \left(-\sqrt{\theta_0}\cdot\Phi^{-1}\left(1-\frac{\alpha}{2}\right), \sqrt{\theta_0}\cdot\Phi^{-1}\left(1-\frac{\alpha}{2}\right)\right) \end{aligned} \tag{1.9}$$

does *not* depend on the actual value of θ_1.

2.b. Assume $0 < \theta_1 < \theta_0$ – The test d_η now reads

$$d_\eta(y) = 0 \quad \text{iff} \quad y^2 > T(\eta;\theta_0,\theta_1), \quad y \in \mathbb{R}.$$

If $\frac{\theta_0}{\theta_1} < \eta^2$, then $\log\left(\eta^2\cdot\frac{\theta_1}{\theta_0}\right) > 0$ but $T(\eta;\theta_0,\theta_1) < 0$. Thus, d_η always selects the null hypothesis H_{θ_0}, whence $\mathbb{P}_{\theta_0}[d_\eta(Y)=1] = 0$ and $\mathbb{P}_{\theta_1}[d_\eta(Y)=1] = 0$.

If $\eta^2 \le \frac{\theta_0}{\theta_1}$, then $\log\left(\eta^2\cdot\frac{\theta_1}{\theta_0}\right) \le 0$ and $T(\eta;\theta_0,\theta_1) \ge 0$, whence

$$d_\eta(y) = 0 \quad \text{iff} \quad |y| > \sqrt{T(\eta;\theta_0,\theta_1)}, \quad y \in \mathbb{R}.$$

It follows that

$$\begin{aligned} \mathbb{P}_{\theta_0}[d_\eta(Y)=1] &= \mathbb{P}_{\theta_0}\left[-\sqrt{T(\eta;\theta_0,\theta_1)} \le Y \le \sqrt{T(\eta;\theta_0,\theta_1)}\right] \\ &= \mathbb{P}_{\theta_0}\left[-\sqrt{\frac{T(\eta;\theta_0,\theta_1)}{\theta_0}} \le \frac{Y}{\sqrt{\theta_0}} \le \sqrt{\frac{T(\eta;\theta_0,\theta_1)}{\theta_0}}\right] \\ &= \Phi\left(\sqrt{\frac{T(\eta;\theta_0,\theta_1)}{\theta_0}}\right) - \Phi\left(-\sqrt{\frac{T(\eta;\theta_0,\theta_1)}{\theta_0}}\right) \\ &= 2\Phi\left(\sqrt{\frac{T(\eta;\theta_0,\theta_1)}{\theta_0}}\right) - 1. \end{aligned} \tag{1.10}$$

Similar calculations show that

$$\mathbb{P}_{\theta_1}\left[d_\eta(Y)=1\right] \quad = \quad 2\Phi\left(\sqrt{\frac{T(\eta;\theta_0,\theta_1)}{\theta_1}}\right)-1. \tag{1.11}$$

Combining the two cases we get

$$\mathbb{P}_{\theta_h}\left[d_\eta(Y)=1\right] = 2\Phi\left(\sqrt{\frac{T(\eta;\theta_0,\theta_1)^+}{\theta_h}}\right)-1, \quad h=0,1. \tag{1.12}$$

To obtain the Neyman-Pearson tests we proceed as follows: Fix α in $(0,1)$. We seek $\eta>0$ such that $\mathbb{P}_{\theta_0}\left[d_\eta(Y)=1\right]=\alpha$. This leads to the equation

$$2\Phi\left(\sqrt{\frac{T(\eta;\theta_0,\theta_1)^+}{\theta_0}}\right)-1=\alpha,$$

or equivalently

$$\Phi\left(\sqrt{\frac{T(\eta;\theta_0,\theta_1)^+}{\theta_0}}\right) = \frac{1+\alpha}{2}.$$

Here we have $\frac{1+\alpha}{2}>\frac{1}{2}$. Any solution is characterized by

$$\frac{T(\eta;\theta_0,\theta_1)^+}{\theta_0} = \left(\Phi^{-1}\left(\frac{1+\alpha}{2}\right)\right)^2,$$

thereby requiring $T(\eta;\theta_0,\theta_1)>0$, namely $<\eta^2<\frac{\theta_0}{\theta_1}$. Therefore, the desired η satisfies

$$\frac{T(\eta;\theta_0,\theta_1)}{\theta_0} = \left(\Phi^{-1}\left(\frac{1+\alpha}{2}\right)\right)^2 \quad \text{and} \quad \eta^2<\frac{\theta_0}{\theta_1}.$$

It is easy to check that any solution to the first equation automatically satisfies the required inequality.

Therefore, the Neyman-Pearson test $d_{\rm NP}(\alpha;\theta_0,\theta_1)$ for testing H_{θ_0} against H_{θ_1} takes the form

$$d_{\rm NP}(\alpha;\theta_0,\theta_1)(y)=0 \quad \text{iff} \quad |y|>\sqrt{\theta_0}\cdot\Phi^{-1}\left(\frac{1+\alpha}{2}\right), \quad y\in\mathbb{R}.$$

Again we note that the set

$$\begin{aligned} C(d_{\rm NP}(\alpha;\theta_0,\theta_1)) &= \left\{y\in\mathbb{R}:\ |y|>\Phi^{-1}\left(\frac{1+\alpha}{2}\right)\cdot\sqrt{\theta_0}\right\} \\ &= \left[-\sqrt{\theta_0}\cdot\Phi^{-1}\left(\frac{1+\alpha}{2}\right),\sqrt{\theta_0}\cdot\Phi^{-1}\left(\frac{1+\alpha}{2}\right)\right]^c \end{aligned} \tag{1.13}$$

does *not* depend on θ_1.

3.

Fix θ_0 and θ_1 so that $0 < \theta_0 < \theta_1$. By Problem **2.a** we know that for each α in $(0,1)$ there exists a Neyman-Pearson test $d_{\rm NP}(\alpha;\theta_0,\theta_1)$ of size α for testing H_{θ_0} against H_{θ_1}; its region is given by

$$C(d_{\rm NP}(\theta_0,\theta_1;\alpha)) = \left(-\sqrt{\theta_0}\cdot\Phi^{-1}\left(1-\frac{\alpha}{2}\right), \sqrt{\theta_0}\cdot\Phi^{-1}\left(1-\frac{\alpha}{2}\right)\right). \tag{1.14}$$

Note that $C(d_{\rm NP}(\theta_0,\theta_1;\alpha))$ does not depend on θ_1 as long as $\theta_0 < \theta_1$.

3.a. With $\Theta_0 = \{1\}$ and $\Theta_1 = (1,\infty)$, it is plain from the remark above that a UMP test $d_{\rm UMP}(\alpha)$ of size α exists for testing $H_0 = H_{\theta_0=1}$ against the composite hypothesis $H_1 \equiv H_\theta,\ \theta \in (1,\infty)$, its region being given by

$$C(d_{\rm NP}(\theta_1,1;\alpha)) = \left(-\Phi^{-1}\left(1-\frac{\alpha}{2}\right), \Phi^{-1}\left(1-\frac{\alpha}{2}\right)\right)$$

Just use $\theta_0 = 1$ in (1.14).

3.c. (We solve Part **3.b.** later). Assume $\Theta_0 = (0,1]$ and $\theta_1 = (1,\infty)$. Pick σ in $(0,1]$. By the arguments given earlier, the UMP test $d_{\rm UMP}(\alpha;\sigma)$ of size α for testing the simple null hypothesis $H_0 \equiv H_\sigma$ against the composite alternative $H_1 \equiv H_\theta,\ \theta \in (1,\infty)$ is given by

$$d_{\rm UMP}(\alpha;\sigma)(y) = 0 \quad \text{iff} \quad |y| < \sqrt{\sigma}\cdot\Phi^{-1}\left(1-\frac{\alpha}{2}\right), \quad y \in \mathbb{R}$$

with region

$$C(d_{\rm UMP}(\alpha;\sigma)) = \left(-\sqrt{\sigma}\cdot\Phi^{-1}\left(1-\frac{\alpha}{2}\right), \sqrt{\sigma}\cdot\Phi^{-1}\left(1-\frac{\alpha}{2}\right)\right).$$

The test $d_{\rm UMP}(\alpha;\sigma)$ is characterized by

$$\mathbb{P}_\theta\left[d(Y)=1\right] \le \mathbb{P}_\theta\left[d_{\rm UMP}(\alpha;\sigma)(Y)=1\right], \qquad \begin{array}{c}\theta > 1\\ d \in \mathcal{D}_{\sigma,\alpha}.\end{array} \tag{1.15}$$

This suggests that by *proper selection* of the parameter σ in $(0,1]$, the test $d_{\rm UMP}(\alpha;\sigma)$ might also be used to implement the UMP test $d_{\rm UMP}(\alpha)$ of size α for testing the composite null hypothesis $H_0 \equiv H_\theta,\ \theta \in (0,1]$ against the composite alternative $H_1 \equiv H_\theta,\ \theta \in (1,\infty)$. Such a test $d_{\rm UMP}(\alpha)$ is characterized by

$$\mathbb{P}_\theta\left[d(Y)=1\right] \le \mathbb{P}_\theta\left[d_{\rm UMP}(\alpha)(Y)=1\right], \qquad \begin{array}{c}\theta > 1\\ d \in \mathcal{D}_{(0,1],\alpha}.\end{array} \tag{1.16}$$

Since $\mathcal{D}_{(0,1],\alpha} \subseteq \mathcal{D}_{\sigma,\alpha}$, it is plain from (1.15) and (1.16) that we can take $d_{\rm UMP}(\alpha) = d_{\rm UMP}(\alpha;\sigma)$ *provided* the test $d_{\rm UMP}(\alpha;\sigma)$ itself is *also* in $\mathcal{D}_{(0,1],\alpha}$.

Thus, we need to answer the following question: Does there exist σ in $(0,1]$ such that

$$\mathbb{P}_{\sigma'}\left[d_{\rm UMP}(\alpha;\sigma)(Y)=1\right] \le \alpha, \qquad 0 < \sigma' \le 1. \tag{1.17}$$

Given σ in $(0,1]$, fix σ' in $(0,1]$. We note that

$$\begin{aligned}
&\mathbb{P}_{\sigma'}\left[d_{\text{UMP}}(\alpha;\sigma)(Y)=1\right] \\
&= \mathbb{P}_{\sigma'}\left[|Y| \geq \sqrt{\sigma}\cdot\Phi^{-1}\left(1-\frac{\alpha}{2}\right)\right] \\
&= \mathbb{P}_{\sigma'}\left[\left|\frac{Y}{\sqrt{\sigma'}}\right| \geq \sqrt{\frac{\sigma}{\sigma'}}\cdot\Phi^{-1}\left(1-\frac{\alpha}{2}\right)\right] \\
&= 1-\mathbb{P}_{\sigma'}\left[\left|\frac{Y}{\sqrt{\sigma'}}\right| < \sqrt{\frac{\sigma}{\sigma'}}\cdot\Phi^{-1}\left(1-\frac{\alpha}{2}\right)\right] \\
&= 1-\left(\Phi\left(\sqrt{\frac{\sigma}{\sigma'}}\cdot\Phi^{-1}\left(1-\frac{\alpha}{2}\right)\right)-\Phi\left(-\sqrt{\frac{\sigma}{\sigma'}}\cdot\Phi^{-1}\left(1-\frac{\alpha}{2}\right)\right)\right) \\
&= F(\sigma;\sigma') \qquad (1.18)
\end{aligned}$$

where we have defined

$$F(\sigma;\sigma') \equiv 2\left(1-\Phi\left(\sqrt{\frac{\sigma}{\sigma'}}\cdot\Phi^{-1}\left(1-\frac{\alpha}{2}\right)\right)\right), \quad \sigma>0, \sigma'>0.$$

Given $\sigma>0$, the mapping $\sigma'\to F(\sigma;\sigma')$ is strictly increasing on $(0,\infty)$ with

$$F(\sigma;\sigma) = 2\left(1-\Phi\left(\Phi^{-1}\left(1-\frac{\alpha}{2}\right)\right)\right) = \alpha.$$

Therefore, the requirement (1.17) that the test $d_{\text{UMP}}(\alpha;\sigma)$ be an element of $\mathcal{D}_{(0,1],\alpha}$ amounts to

$$\sup_{\sigma'\in(0,1]}\left(\mathbb{P}_{\sigma'}\left[d_{\text{UMP}}(\alpha;\sigma)(Y)=1\right]\right) \leq \alpha, \qquad (1.19)$$

or equivalently

$$\sup_{\sigma'\in(0,1]}\left(2\left(1-\Phi\left(\sqrt{\frac{\sigma}{\sigma'}}\cdot\Phi^{-1}\left(1-\frac{\alpha}{2}\right)\right)\right)\right) \leq \alpha.$$

But strict monotonicity and continuity imply that

$$\sup_{\sigma'\in(0,1]}\left(2\left(1-\Phi\left(\sqrt{\frac{\sigma}{\sigma'}}\cdot\Phi^{-1}\left(1-\frac{\alpha}{2}\right)\right)\right)\right) = 2\left(1-\Phi\left(\sqrt{\sigma}\cdot\Phi^{-1}\left(1-\frac{\alpha}{2}\right)\right)\right)$$

(with the supremum achieved at $\sigma'=1$) and the question reduces to finding σ in $(0,1]$ such that

$$2\left(1-\Phi\left(\sqrt{\sigma}\cdot\Phi^{-1}\left(1-\frac{\alpha}{2}\right)\right)\right) \leq \alpha. \qquad (1.20)$$

This constraint is equivalent to

$$1-\frac{\alpha}{2} \leq \Phi\left(\sqrt{\sigma}\cdot\Phi^{-1}\left(1-\frac{\alpha}{2}\right)\right),$$

hence $\sigma \geq 1$ by mononotonicity! This shows that $d_{\text{UMP}}(\alpha;\sigma)$ with $\sigma=1$ satisfies the requirement (1.17) and $d_{\text{UMP}}(\alpha) = d_{\text{UMP}}(\alpha;1)$ as desired!

3.b. We now turn to the case $\Theta_0 = (0,1)$ and $\theta_1 = (1,\infty)$. Recall from Part **3.c** that the test $d_{\rm UMP}(\alpha;1)$ is characterized by

$$\mathbb{P}_\theta\left[d(Y)=1\right] \leq \mathbb{P}_\theta\left[d_{\rm UMP}(\alpha;1)(Y)=1\right], \qquad \begin{array}{c}\theta>1\\ d\in\mathcal{D}_{(0,1],\alpha}.\end{array} \tag{1.21}$$

On the other hand, the desired UMP test $d_{\rm UMP}(\alpha)$ of size α that tests the composite null hypothesis $H_0 \equiv H_\theta,\ \theta\in(0,1)$ against the composite alternative $H_1 \equiv H_\theta,\ \theta\in(1,\infty)$ is characterized by the inequalities

$$\mathbb{P}_\theta\left[d(Y)=1\right] \leq \mathbb{P}_\theta\left[d_{\rm UMP}(\alpha)(Y)=1\right], \qquad \begin{array}{c}\theta>1\\ d\in\mathcal{D}_{(0,1),\alpha}.\end{array} \tag{1.22}$$

It is of course tempting to conjecture that $d_{\rm UMP}(\alpha) = d_{\rm UMP}(\alpha;1)$ here as well – However, beware of the inclusion $\mathcal{D}_{(0,1],\alpha} \subseteq \mathcal{D}_{(0,1),\alpha}$! Nevertheless we see that this conjecture will hold if we show that (i) the equality $\mathcal{D}_{(0,1],\alpha} = \mathcal{D}_{(0,1),\alpha}$ holds (although we have the inclusion $\mathcal{D}_{(0,1],\alpha} \subseteq \mathcal{D}_{(0,1),\alpha}$) and that (ii) the inequalities (1.22) are implied by the inequalities (1.21)! A moment of reflection should convince you that only (i) needs to be established, and that this equality is an easy immediate consequence of the following fact.

For each d in $\mathcal{D}$, the mapping $(0,\infty)\to[0,1]:\theta\to\mathbb{P}_\theta\left[d(Y)=1\right]$ is continuous.

This can be shown as follows. Pick d in $\mathcal{D}$ and $\theta>0$. For every $\theta'>0$ note that

$$\mathbb{P}_{\theta'}\left[d(Y)=1\right] = \int_{C(d)^c} f_{\theta'}(y)dy.$$

With $\varepsilon>0$ in $(0,\theta)$ and θ' in $(\theta-\varepsilon,\theta+\varepsilon)$ we have the following obvious inequalities:

$$\begin{aligned} f_{\theta'}(y) &= \frac{1}{\sqrt{2\pi\theta'}}e^{-\frac{y^2}{2\theta'}} \\ &\leq \frac{1}{\sqrt{2\pi\theta'}}e^{-\frac{y^2}{2(\theta+\varepsilon)}} \\ &\leq \frac{1}{\sqrt{2\pi(\theta-\varepsilon)}}e^{-\frac{y^2}{2(\theta+\varepsilon)}} \\ &= \sqrt{\frac{\theta+\varepsilon}{\theta-\varepsilon}}\cdot f_{\theta+\varepsilon}(y), \quad y\in\mathbb{R} \end{aligned} \tag{1.23}$$

with

$$\int_{\mathbb{R}} f_{\theta+\varepsilon}(y)dy = 1.$$

It follows from the Dominated Convergence Theorem that

$$\lim_{n\to\infty}\int_{C(d)^c} f_{\theta_n}(y)dy = \int_{C(d)^c}\lim_{n\to\infty} f_{\theta_n}(y)dy = \int_{C(d)^c} f_\theta(y)dy$$

for any any sequence $\mathbb{N}_0 \to (0,1) : n \to \theta_n$ such that $\lim_{n\to\infty} \theta_n = \theta$. This establishes the desired continuity.

4.

4.a. Fix d in $\mathcal{D}$. For each $\boldsymbol{p}$ in $\mathcal{P}_M$, we note that

$$\begin{aligned} J_{\boldsymbol{p}}(d) &= \mathbb{E}_{\boldsymbol{p}}\left[C(H, d(\boldsymbol{Y}))\right] \\ &= \sum_{m=0}^{M-1} p_m \mathbb{E}_{\boldsymbol{p}}\left[C(H, d(\boldsymbol{Y})) | H = m\right] \\ &= \sum_{m=0}^{M-1} p_m \mathbb{E}_{\boldsymbol{p}}\left[C(m, d(\boldsymbol{Y})) | H = m\right] \end{aligned} \tag{1.24}$$

with the quantities

$$\mathbb{E}_{\boldsymbol{p}}\left[C(m, d(\boldsymbol{Y})) | H = m\right], \quad m = 0, 1, \ldots, M-1$$

independent of $\boldsymbol{p}$. For arbitrary $\boldsymbol{p}_0$ and $\boldsymbol{p}_1$ in $\mathcal{P}_M$ it is now plain that

$$J_{\lambda \boldsymbol{p}_1 + (1-\lambda)\boldsymbol{p}_0}(d) = \lambda J_{\boldsymbol{p}_1}(d) + (1-\lambda) J_{\boldsymbol{p}_0}(d), \quad \lambda \in (0,1).$$

Now recall that

$$V(\boldsymbol{p}) \equiv \inf_{d \in \mathcal{D}} J_{\boldsymbol{p}}(d), \quad \boldsymbol{p} \in \mathcal{P}_M$$

so that

$$V(\boldsymbol{p}) \leq J_{\boldsymbol{p}}(d), \quad \begin{array}{c} d \in \mathcal{D} \\ \boldsymbol{p} \in \mathcal{P}_M. \end{array}$$

Therefore, fix d in $\mathcal{D}$. For arbitrary $\boldsymbol{p}_0$ and $\boldsymbol{p}_1$ in $\mathcal{P}_M$ we get

$$\begin{aligned} J_{\lambda \boldsymbol{p}_1 + (1-\lambda)\boldsymbol{p}_0}(d) &= \lambda J_{\boldsymbol{p}_1}(d) + (1-\lambda) J_{\boldsymbol{p}_0}(d) \\ &\geq \lambda V(\boldsymbol{p}_1) + (1-\lambda) V(\boldsymbol{p}_0), \quad \lambda \in (0,1). \end{aligned} \tag{1.25}$$

and the conclusion

$$\lambda V(\boldsymbol{p}_1) + (1-\lambda) V(\boldsymbol{p}_0) \leq V(\lambda \boldsymbol{p}_1 + (1-\lambda)\boldsymbol{p}_0), \quad \lambda \in (0,1)$$

follows.

4.b. We have

$$\begin{aligned} J_{\boldsymbol{p}^\star}(d^\star) &= J_{\boldsymbol{p}}(d^\star) \quad \begin{array}{l} [\text{For all } \boldsymbol{p} \text{ in } \mathcal{P}_M \text{ since} \\ \text{the mapping } \boldsymbol{p} \to J_{\boldsymbol{p}}(d^\star) \text{ is constant}] \end{array} \\ &= J_{\text{Max}}(d^\star) \\ &\geq \inf_{d \in \mathcal{D}} J_{\text{Max}}(d) \\ &= \inf_{d \in \mathcal{D}} \left(\sup_{\boldsymbol{p} \in \mathcal{P}_M} J_{\boldsymbol{p}}(d) \right) \\ &\geq \inf_{d \in \mathcal{D}} J_{\boldsymbol{p}^\star}(d) \\ &= V(\boldsymbol{p}^\star). \end{aligned} \tag{1.26}$$

Because $V(\boldsymbol{p}^\star) = J_{\boldsymbol{p}^\star}(d^\star)$ we conclude that

$$J_{\text{Max}}(d^\star) = \inf_{d \in \mathcal{D}} J_{\text{Max}}(d),$$

and $d^\star$ is indeed a minimax strategy.

4.c. The assumption $V(\boldsymbol{p}^\star) = J_{\boldsymbol{p}^\star}(d^\star)$ immediately yields

$$V(\boldsymbol{p}^\star) \geq \inf_{d \in \mathcal{D}} \left(\sup_{\boldsymbol{p} \in \mathcal{P}_M} J_{\boldsymbol{p}}(d) \right) \tag{1.27}$$

by virtue of (1.26). It follows that

$$\inf_{d \in \mathcal{D}} \left(\sup_{\boldsymbol{p} \in \mathcal{P}_M} J_{\boldsymbol{p}}(d) \right) \leq V(\boldsymbol{p}^\star) \leq \sup_{\boldsymbol{p} \in \mathcal{P}_M} V(\boldsymbol{p}) = \sup_{\boldsymbol{p} \in \mathcal{P}_M} \left(\inf_{d \in \mathcal{D}} J_{\boldsymbol{p}}(d) \right). \tag{1.28}$$

On the other hand the inequality

$$\sup_{\boldsymbol{p} \in \mathcal{P}_M} V(\boldsymbol{p}) \leq \inf_{d \in \mathcal{D}} \left(\sup_{\boldsymbol{p} \in \mathcal{P}_M} J_{\boldsymbol{p}}(d) \right) \tag{1.29}$$

always holds as a result of the obvious inequalities

$$V(\boldsymbol{p}) \leq J_{\boldsymbol{p}}(d) \leq J_{\text{Max}}(d), \qquad \begin{array}{c} \boldsymbol{p} \in \mathcal{P}_M \\ d \in \mathcal{D}. \end{array}$$

The proof is as in the binary case.

Combining the inequalities (1.28) and (1.29) yields the Minimax Equality

$$\sup_{\boldsymbol{p} \in \mathcal{P}_M} \left(\inf_{d \in \mathcal{D}} J_{\boldsymbol{p}}(d) \right) = \inf_{d \in \mathcal{D}} \left(\sup_{\boldsymbol{p} \in \mathcal{P}_M} J_{\boldsymbol{p}}(d) \right) \tag{1.30}$$

as well as the fact that

$$V(\boldsymbol{p}^\star) = \sup_{\boldsymbol{p} \in \mathcal{P}_M} V(\boldsymbol{p}).$$

In short, $\boldsymbol{p}^\star$ is a maximum of the mapping $\mathcal{P}_M \to \mathbb{R} : \boldsymbol{p} \to V(\boldsymbol{p})$.