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INTERCONNECTION NETWORKS
ANALYSIS DESIGN and ROUTING

By

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Part II

UNORDERED CONNECTIONS
INTRODUCTION

This part is devoted to the study of connector design problems in which the order of connections is considered not critical. In set-theoretic terms, such connectors or networks realize certain one-to-one and/or many-to-one maps between an $n$-set of inputs and an $m$-set of indistinguishable outputs. These can broadly be divided into two classes: concentration and distribution. Concentration problems entail constructions of connectors that permit a set of one-to-one connections between their inputs and outputs with some specificity about the maximum number and/or relative locations of the outputs that can take part in the connections. Distribution (sometimes, called generalization) problems deal with constructions of connectors that permit a set of one-to-many connections between their inputs and outputs, also with some specificity about the maximum number and relative locations of outputs. In some sense, concentration is the opposite of distribution and this fact will be useful in designing distribution networks from concentrators. Furthermore, these two basic types of connectors can be refined to obtain other unordered connectors such as strong concentrators, superconcentrators, generalizers, and expanders.

In keeping with the main goals of the book, the flavor of the problems we encounter in this part will be questions such as how much concentration, or expansion one can get for a given number of crosspoints, or how many crosspoints it takes to construct a concentrator or an expander. We will also tackle the constructions of such connectors within constant factors of lower crosspoint, depth, and degree bounds.

In pursuit of these questions, much of our attention throughout this part will center on a notion, called capacity. The capacity of a concentrator or an expander determines the number of calls or connections that can proceed in parallel. Concentrators that permit as many arbitrary parallel calls as they have outputs are called full capacity connectors, and those that permit fewer parallel calls are called bounded capacity connectors. In general the more crosspoints a concentrator encompasses, the larger is its capacity provided that
its crosspoints are intelligently distributed. It is this design flavor that makes the study of concentrators an interesting subject.

In a nutshell, our main problem is the following: Given a fixed number of crosspoints, a set of inputs, and a set of outputs; place the crosspoints between the inputs and outputs so that the capacity of the resulting connector is maximized. We will refine this problem in several directions. One direction is to consider networks in which all connections proceed in space, another direction will be to consider networks that permit buffering so that connections can be made in time, and yet another alternative is to combine these two approaches together. As we will see, these will lead to space division, time division, and space-and-time division networks. Our primary focus will be on space-division networks, but we will consider some space-and-time division network problems as well.

This part is organized as follows. We begin in Chapter 2 with the formalization of basic concentrator notions in terms of matching sets in directed multipartite graphs. We then derive lower bounds on the vertex, edge, crosspoint, diameter, and depth complexities of concentrators. Next, we give a lower bound on the crosspoint complexity of sparse crossbar (bipartite) concentrators. We prove that this bound is tight by presenting a number of optimal concentrator designs. We also show how to balance the fanin and fanout of sparse crossbar concentrators without altering their optimality. We then extend these results to bounded capacity concentrators. Next, we introduce a weaker notion of capacity, called the deficiency of a concentrator. We invoke graph theoretical facts such as Hall and König theorems to compute the deficiency of a concentrator. We conclude Chapter 2 with the examination of equivalence relations among sparse crossbar concentrators and a basic result on the density of sparse crossbar concentrators among all the $2^{nm}$ sparse crossbars with $n$ inputs and $m$ outputs.

In Chapter 3, we focus on the following question: Does there exist a concentrator whose crosspoint complexity increases linearly with its number of inputs? We will find out that the answer is in the affirmative, but its conclusion requires an ingenious combinatorial argument due to Pinsker. We give a detailed account of Pinsker’s original work
on this question. We also present Bassalygo’s and Chu’s extensions on the existence of concentrators with linear crosspoint complexity. Natural extensions of these constructions to superconcentrators and generalizers will also be given in this chapter.

In Chapter 4, we switch gears to tackle the construction of concentrators with linear crosspoint complexity head on. We will see that the solution of this problem is closely related to the existence of a square-shaped expander with linear crosspoint complexity. We will give an in depth account of Margulis’ seminal work on expanders. Our coverage will also include some of the more recent constructions of bounded capacity concentrators and expanders with linear crosspoint complexity including Gabber and Galil’s and Alon’s expanders.

In Chapter 5, we will see how bounded capacity concentrators can be cascaded together to obtain concentrators, hyperconcentrators, strong concentrators, and superconcentrators, all with linear crosspoint complexity.

Finally, in Chapter 6, we describe how distributors and generalizers can be constructed using concentrators and expanders.
Chapter 2

CONCENTRATORS

“It is not book learning young men need, nor instruction about this and that, but a stiffening of the vertebrae which will cause them to concentrate their energies do a thing, carry a message to Garcia.”

ELBERT HUBBARD (1856-1915), A message to Garcia, (March 1899)

2.1 INTRODUCTION

Dictionaries define the word “concentration” as the act of bringing things to a common center or small space. Passengers boarding an airplane, spectators exiting a theater, and vehicles merging towards a single lane on a multilane road are all examples of a concentration process. Electrical signals carrying raw data, digitized voice and video packets are transmitted from one location to another much the same way over multiplexed lines. This concentration or multiplexing process takes on a variety of forms depending upon the particular transmission protocol used, and the technology that is available. If signals are transmitted over a single cable or carrier, the transmission protocol typically relies on dividing the available bandwidth into time slots or channels as in time or frequency division multiplexing, but the same transmission protocols may be used in multi-cable and multi-carrier systems as well.
Nonetheless, the network backbone of a single-cable concentrators need not be more than a simple multiplexer, whereas the design of multi-cable concentrators requires a more careful examination of the tradeoffs available between time, space and available cable bandwidth. For example, consider a hypothetical 6-channel 2-cable video movie distribution system. If each cable has sufficient bandwidth to carry three channels, then the six channels can be divided into two sets of three channels and each set of channels can then be transmitted through a multiplexer as shown in Figure 2.1(a). With this set up, each customer can concurrently view up to six channels. Now suppose that the cable company wants to enhance the picture quality by transmitting only two programs over each cable. In this case, a customer can view at most four channels concurrently, but the multiplexing set-up in Figure 2.1(a) adds one more restriction: Among the fifteen choices of four channels out of the six channels, only nine \((\binom{3}{2}\binom{3}{2})\) can be viewed concurrently by a customer. In particular, any combination of four channels three of which are assigned to the same multiplexer cannot be viewed concurrently by a customer.

We can alleviate this problem by first concentrating the six channels onto four intermediate channels, and then multiplexing the two of the four intermediate channels into one cable and multiplexing the other two onto another cable as shown in Figure 2.1(b). Now, any two of the six channels can be multiplexed onto one cable, and any two of the remaining four channels can be multiplexed to the other cable, allowing each customer to receive any four of the six channels.

The first part of this multiplexing scheme is generally referred to as a space division concentrator to distinguish it from time or frequency division multiplexing, and to emphasize that channels are separated in space rather than in time or frequency. Once channels are separated in space they can then be transmitted by a time-division or frequency division multiplexer. In Figure 2.1(b), it is assumed that the channels share time as a resource; time is divided into a fixed number of slots, where each time slot carries one channel in a periodic fashion.

In this and next four chapters, we will focus our attention on the space division aspect of concentration problems. We will deal with
Figure 2.1: Multiplexing television channels to a pair of cables. In (a), the channels are divided into two groups and each group is time-multiplexed onto a cable. In (b), the channels are first concentrated onto four intermediate channels; intermediate channels are then divided into two groups, and finally, each group of intermediate channels is time-multiplexed onto a cable.

time multiplexing issues in the second part of the text when we examine potential tradeoffs between switching cost and transmission time in concentrator designs.

Loosely speaking, a space-division concentrator (or simply concentrator) is a switching network with $n$ inputs and $m \leq n$ outputs in which there exist non-overlapping paths between any $k \leq m$ of the $n$ inputs and some $k$ of the $m$ outputs.\footnote{Concentrators customarily have fewer outputs than inputs. While this makes sense in practical terms, including $m = n$ case in our definition allows an easier classification of various types of concentrators.} As in any system design, the design of a concentrator is determined by the desired performance and cost. In the above example, we must use enough cables to meet customer service requirements. The number of cables in turn determines the complexity of the concentrator that must be placed between the channels and cables. Furthermore, in certain situations, a stronger notion of concentration may be needed. We may require any given $k$ inputs to reach consecutive $k$ outputs, or even every $k$ outputs, rather than just some $k$ outputs for a specified range of values of $k$. 

As in any system design, the design of a concentrator is determined by the desired performance and cost. In the above example, we must use enough cables to meet customer service requirements. The number of cables in turn determines the complexity of the concentrator that must be placed between the channels and cables. Furthermore, in certain situations, a stronger notion of concentration may be needed. We may require any given $k$ inputs to reach consecutive $k$ outputs, or even every $k$ outputs, rather than just some $k$ outputs for a specified range of values of $k$.\footnote{Concentrators customarily have fewer outputs than inputs. While this makes sense in practical terms, including $m = n$ case in our definition allows an easier classification of various types of concentrators.}
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We begin this chapter with the formalization of these ideas in a graph theoretical setting, and derivation of some elementary bounds on the switching complexity of concentrators. We then present various sparse crossbar realizations of concentrators spanning a range of constructions that meet key design constraints such as crosspoint complexity, fanin, and fanout. Finally, we examine the structural relations among sparse crossbar concentrators, and investigate the density of sparse crossbar concentrators among all sparse crossbars with a fixed number of inputs and outputs.

2.2 DEFINITIONS

The notion of a concentrator ties with matching sets in graphs. A directed graph $G$ with $n$ source vertices (inputs) and $m$ sink vertices (outputs), $1 \leq m \leq n$, is called an $(n, m)$-concentrator, if every $k$ of the $n$ source vertices, $1 \leq k \leq m$, has a matching set among the $m$ outputs of $G$. A matching is to be viewed as a set of edge-disjoint paths between inputs and outputs; vertices may be shared among paths provided that the number of paths entering a vertex is equal to the number of paths leaving that vertex.

Figure 2.2 depicts a $(4,3)$-graph, and illustrates how two inputs can be matched with two outputs over edge-disjoint paths (dark edges and vertices). It can be verified that any 1, 2, and 3 inputs can be matched with some 1, 2 and 3 outputs in this graph so it is a $(4,3)$-concentrator. However, it is important to emphasize that the individual outputs in a matching cannot be dictated. For example, in Figure 2.2, the two inputs, i.e., $a_2$ and $a_4$ cannot be matched with the pairs of outputs $b_1, b_2$ or $b_2, b_3$. To see that this is not a very stringent limitation, consider a set of remote computer users who seek to login to a main server. Users are ordinarily connected to a server by a set of I/O processors, but they need not specify the actual I/O processor in the connection. This is exactly what an $(n, m)$-concentrator provides, where $n$ is the number of users and $m$ is the number of I/O processors.

In other concentration problems, it may be critical to distinguish among groups of outputs, even if not between individual outputs.
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Figure 2.2: The graph structure of a concentrator. The labeled vertices represent inputs and outputs. The remaining vertices represent switches. The darker edges and vertices form a matching between inputs $a_2, a_4$ and outputs $b_1, b_3$.

within each group. For example, if five computers and five printers are to be shared among twenty users, then a simple $(20,10)$-concentrator is not sufficient to permit every ten users to simultaneously access a particular distribution of computers and printers. In general, the degree of uncertainty in concentrating a set of inputs to a set of outputs hints at stronger forms of concentration.

**Definition 2.1** An $(n,m)$-concentrator is called a strong $(n,m)$-concentrator if there exist subsets of outputs, $Y_1, Y_2, \ldots, Y_m$, where $|Y_k| = k, 1 \leq k \leq m$, such that $Y_k$ is a matching set of every $k$ inputs, $1 \leq k \leq m$. ||
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This definition implies that all subsets of inputs of any given cardinality in a strong concentrator have a common matching set. That is, all \( k \)-subsets of inputs can be concentrated to the same subset of \( k \) outputs, \( 1 \leq k \leq m \). This fact disqualifies the \((4,3)\)-concentrator in Figure 2.2 from being a strong concentrator since inputs \( a_1, a_2 \) cannot be matched to \( b_1, b_2 \) while inputs \( a_1, a_3 \) cannot be matched to \( b_1, b_3 \) or \( b_2, b_3 \). Furthermore, it can be shown that this property also implies that the matching set of any \( k \)-subset of inputs contain the matching set of any \( k' \)-subset of inputs for all \( k' < k \). This latter property can be strengthened by using even a more powerful concentrator.

**Definition 2.2** An \((n,m)\)-concentrator is called an \((n,m)\)-superconcentrator if every \( k \)-subset of its outputs is a matching set of every \( k \)-subset of its inputs, \( 1 \leq k \leq m \).

Unlike concentrators and strong concentrators, superconcentrators can continue to provide matchings between their inputs and outputs when some of the inputs and outputs fail to function properly. This can be pivotal in a graceful degradation of performance in many communication systems. For example, in a cable television system with replicated channels over a superconcentrator, customers can continue to receive service as long as some fixed number of the replicated channels continue to function. Also, in a superconcentrator, the inputs can partially specify their outputs. For example, consider the graph in Figure 2.3. It can be verified that this is a \((4,4)\)-superconcentrator. Furthermore, the inputs can designate their outputs in some cases. For example, \( a_1, a_2 \) can be matched with each pair of outputs so that \( a_1 \) is mapped to either of the outputs, and \( a_2 \) is mapped to the other input. On the other hand, it is not always possible to connect the inputs to the outputs in a desired order. For example, the same graph cannot realize the mapping \( a_1 \rightarrow b_1, a_2 \rightarrow b_4, a_3 \rightarrow b_3, a_4 \rightarrow b_2 \). This is because, both \( a_2 \) and \( a_4 \) must use the edge between \( B \) and \( D \) to reach their outputs. The matching can surely be achieved if we relax the ordering on the outputs. Likewise, \( a_1 \) and \( a_3 \) can be mapped only in one way to \( b_1 \) and \( b_3 \), since mapping \( a_1 \) to \( b_3 \) and \( a_3 \) to \( b_1 \) requires that the edge between \( A \) and \( C \) be shared. In fact, this graph can realize only 18 of the 24 bijections between its inputs and outputs.
In all of the above concentrators we assume that the number of inputs that can be matched may run up to the total number of outputs. In certain concentration problems, the statistical (or even deterministic) behavior of connection requests may dictate that the cardinality of matching sets rarely (or never) exceed a certain threshold. As an example, suppose that a computer laboratory has $m$ workstations that are logged in by no more than $c$ of some $n$ remote users at any given time, where $c \leq m \leq n$. In this case, the connections between the remote users and workstations can be defined as a bounded capacity concentration problem with varying degrees of specificity of outputs as follows.

**Definition 2.3** A graph $G$ with $n$ inputs and $m$ outputs is called

(a) an $(n,m,c)$-concentrator if any $k$ of its inputs has a matching set, for any $k$, $1 \leq k \leq c$;

(b) a strong $(n,m,c)$-concentrator if there exist subsets of outputs, $Y_1, Y_2, \ldots, Y_c$, where $|Y_k| = k$, $1 \leq k \leq c$, such that $Y_k$ is a matching set of every $k$ inputs, $1 \leq k \leq c$;

(c) an $(n,m,c)$-superconcentrator if every $k$ subset of its outputs is a matching set of every $k$ subset of its inputs, $1 \leq k \leq c$. 

![Figure 2.3: A (4,4)-superconcentrator.](image-url)
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The coefficient $c$ is called the capacity of $G$. ||

The following propositions are easily proved.

**Proposition 2.1** Every $(n, m, c)$-concentrator (strong concentrator, superconcentrator) is an $(n, m, c')$-concentrator (strong concentrator, superconcentrator) for all $n \geq m \geq c \geq c'$.

**Proposition 2.2** Every $(n, m, c)$-superconcentrator is an $(n, r, c)$-superconcentrator for all $n \geq m \geq r \geq c$.

**Proposition 2.3**
(a) Every $(n, m)$-strong concentrator is an $(n, m)$-concentrator, but the converse is not true.
(b) Every $(n, m)$-superconcentrator is an $(n, m)$-strong concentrator, but the converse is not true.

Interestingly enough, the analog of Proposition 2.2 does not necessarily hold for an $(n, m, c)$-concentrator, or even for a strong $(n, m, c)$-concentrator. To see this, consider an $(n, n - 1, c)$-concentrator whose $i$th input is connected to its $i$th output, $1 \leq i \leq n - 1$ and its $n$th input is connected to all of its outputs. No matter which output is removed, the resulting graph cannot be an $(n, n - 2, c)$-concentrator since the removal of any output leaves one of the first $n - 1$ inputs floating, i.e., unconnected to any output. For example, in Figure 2.2, if any of the outputs is removed then one of $a_1, a_2, a_3, a_4$ will float and force the graph to have a zero capacity. Likewise, it can be shown that not every strong $(n, m, c)$-concentrator is a strong $(n, r, c)$-concentrator for $m > r \geq c$.

The relations between the concentrator classes introduced so far are shown in Figure 2.4. All concentrators in this representation have the same number of inputs and outputs, and have the same capacity in the case of bounded concentrators. The arrows indicate the circles (pairs of circles) which bound each class of concentrators. It should be easy to see that the set of bounded concentrators contains all other concentrators. Likewise, the set of all concentrators contains the set of all strong concentrators which in turn contains the set of all superconcentrators. It may be useful to point out that bounded capacity
DEFINITIONS

superconcentrators do not degenerate to full capacity concentrators except when they are bipartite. To see this, consider the graph in Figure 2.5. By inspection, it is easily seen that there is a path between every input and every output, but there are no edge-disjoint paths between $a_1, a_2$ and $b_3, b_4$. Likewise, $a_3, a_4$ cannot be matched with $b_1, b_2$. Hence this graph is a $(4, 4, 1)$-superconcentrator, but it is not a $(4, 4, 2)$-superconcentrator. A similar example can be given for strong concentrators as well. It should also be noted that the set of strong concentrators can further be refined to account for the possibility of having bounded strong concentrators with different capacities which are also concentrators. In any case, this classification is only representative of the concentrators reported in the literature, and other types of concentrators can be defined and added to this classification.

2.2.1 Complexity of Concentration Problems

Before we attempt to construct the various concentrators defined in the previous section, it is worthwhile to determine their vertex, edge,
Figure 2.5: A graph which is a $(4, 4, 1)$-superconcentrator, but not a $(4, 4, 2)$-superconcentrator.

diameter and depth complexities.

Let $\mathcal{G}(n, m, c)$ be a $(n, m, c)$-concentrator or strong concentrator with $q_e$ vertices each with $x$ inputs and $y$ outputs. Each vertex can realize at most $(1 + y)^x$ maps, and hence the number of distinct maps that $\mathcal{G}(n, m, c)$ can realize does not exceed $(1 + y)^{xq_e}$. Now $\mathcal{G}$ must realize any one of $\sum_{i=0}^c \binom{m_i}{i}$ distinct matchings between its $n$ inputs and $m$ outputs. It follows that

$$
(1 + y)^{xq_e} \geq \sum_{i=0}^c \binom{n}{i},
$$
or the number of vertices in an $(n, m, c)$-concentrator (or strong concentrator) must satisfy the inequality

$$
q_e \geq \frac{\log \left(\sum_{i=0}^c \binom{n}{i}\right)}{x \log(1 + y)}.
$$

(2.1)

Using the same argument, we find that the number of vertices $q_s$ in an $(n, m, c)$-superconcentrator must satisfy the inequality

$$
q_s \geq \frac{\log \left(\sum_{i=0}^c \binom{n}{i} \binom{m}{i}\right)}{x \log(1 + y)}.
$$

(2.2)
These inequalities indicate that when \( c = m = n \),

\[
q_c, q_s \geq \frac{n}{x \log(1 + y)},
\]

and hence the vertex complexity of concentrators and superconcentrators is lower bounded by a term which is not greater than \( n \).

Alternatively, we can derive a graph theoretical bound as follows. With each output vertex having \( x \) inputs, the \( n \) inputs must be fanned out to at least \( n/x \) vertices. Likewise, with each input vertex having \( y \) outputs, the \( m \) outputs must be fanned in to at least \( m/y \) vertices. Hence, we obtain the following tighter lower bounds

\[
q_c \geq \frac{n}{x}, \text{ and } q_s \geq \text{Max}(n/x, m/y).
\]

Given a lower bound on the vertex complexity of an \((n, m, c)\)-concentrator, and no other information about the fanin and fanout of individual switching vertices, it can at most be asserted that its crosspoint complexity matches its vertex complexity. However, if each switching vertex has a fanin \( x \) and fanout \( y \), and it is realized by a complete bipartite graph, then multiplying the lower bounds on vertex count by \( xy \) yields the following lower bounds on the crosspoint complexity of concentrators and superconcentrators

\[
\kappa_c \geq ny, \text{ and, } \kappa_s \geq \text{Max}\{ny, mx\}.
\]

Similarly, under the same assumption, and multiplying the vertex complexities by \( x + y \) lower bounds the edge-count of concentrators and superconcentrators;

\[
\xi_c \geq n(1 + \frac{y}{x}), \text{ and } \xi_s \geq \text{Max}\{n(1 + \frac{y}{x}), m(1 + \frac{x}{y})\}.
\]

These inequalities indicate that the crosspoint complexity of concentrators and superconcentrators is at least \( n \), and their edge complexity is at least \( 2n \) when \( x = y \).

As for diameter, there is no obvious constraint other than fanin and fanout restrictions that leads to a meaningful lower bound for concentrators. Hence an \((n, m)\)-concentrator has a trivial diameter lower
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bound of $\Omega(1)$. In the case of a superconcentrator, each input must be connected to each output, and hence the diameter of an $(n, m)$-superconcentrator constructed out of switching vertices with $x$ inputs and $y$ outputs must satisfy

$$
\rho_s \geq \text{Max}\{\log_x n, \log_y m\}.
$$

(2.7)

The depth lower bounds are obtained by multiplying the diameter lower bounds with $\log x + \log y$.

2.2.2 Constant-Degree Concentrators

While the lower bounds on concentrators are stated for arbitrary fanin and fanout, technological factors often dictate that connectors be realized using switching vertices with constant fanin and fanout. Concentrators are no exception in this regard.

The most obvious way to enforce this constraint is to use trees with constant fanin and fanout. For example, a strong $(n, m)$-concentrator can be constructed by first coalescing the leaf vertices of $n$ copies of an $(2m - 1)$-vertex binary tree with the leaf vertices of $m$ copies of an $(2n - 1)$-vertex binary tree and then removing the coalesced vertices. Exactly two leaf vertices are coalesced between every pair of trees, one from each set as illustrated in Figure 2.6(a) for $n = 4$, and $m = 3$.

It can be shown that the strong $(n, m)$-concentrator obtained this way has

(a) $n(m - 1) + m(n - 1)$ vertices,

(b) $2[n(m - 1) + m(n - 1)]$ crosspoints,

(c) $2[n(m - 1) + m(n - 1)]$ edges,

(d) $\lceil \log n \rceil + \lceil \log m \rceil - 1$ diameter and depth.

The vertex, edge and crosspoint complexities can be reduced by noting that not all inputs need be connected to all outputs. Let $a_i, 1 \leq i \leq n$ and $b_i, 1 \leq i \leq m$, denote the input and output vertices of the coalesced binary trees. The paths between input $a_i$ and
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Figure 2.6: Strong (4,3)-concentrators. (Dashed circles and lines indicate redundant vertices and edges.)
outputs $b_{i+1}, b_{i+2}, \ldots, b_m, 1 \leq i \leq m - 1$, can all be removed as illustrated in Figure 2.6(b) for $n = 4$ and $m = 3$. This leaves the paths between input $a_i$ and outputs $b_1, b_2, \ldots, b_i, 1 \leq i \leq m$ and those between inputs $a_i, m + 1 \leq i \leq n$ and outputs $b_1, b_2, \ldots, b_m$ intact.

Let $G$ denote the modified graph, and let us renumber the inputs of $G$ by $1, 2, \ldots, n$. That $G$ is a strong concentrator can be proved as follows. Let $x_1, x_2, \ldots, x_\alpha$, where $1 \leq x_1 < x_2 < \ldots < x_\alpha \leq n$, denote any $\alpha \geq 1$ of the $n$ inputs. Obviously, $x_\alpha \geq \alpha, x_{\alpha-1} \geq \alpha - 1, \ldots x_1 \geq 1$. Therefore, by the construction of $G$ there exist edge-disjoint paths between pairs of inputs and outputs $(x_1, b_1), (x_2, b_2), \ldots, (x_\alpha, b_\alpha)$, that is, $\{b_1, b_2, \ldots, b_\alpha\}$ is a matching set of any $\alpha \geq 1$ inputs. Thus, it follows by Definition 2.1 that $G$ is a strong $(n, m)$-concentrator.

The diameter and depth of this modified strong $(n, m)$-concentrator remain the same as those in the original construction. Its vertex, crosspoint and edge complexities are reduced, and they can be determined by noting that

1. input $a_i$ is connected by a path to $i$ outputs when $1 \leq i \leq m$, and to $m$ outputs when $m + 1 \leq i \leq n$,
2. output $b_i$ is connected by a path to $n - i + 1$ inputs, $1 \leq i \leq m$.

If the inputs and outputs are connected by binary trees as in Figure 2.6(b), then (1) implies that fanning out the inputs to the leaf vertices of the binary trees at the top requires

$$\sum_{i=2}^{m} (i - 1) + (n - m)(m - 1) = (m - 1)(n - (m/2))$$

vertices, and fanning out the outputs to the leaf vertices of the binary trees at the bottom requires

$$\sum_{i=1}^{m} (n - i) = m(n - (m + 1)/2)$$

vertices. It follows that the modified strong concentrator uses

$$(m - 1)(n - (m/2)) + m(n - (m + 1)/2)$$

vertices, $2\{(m - 1)(n - (m/2)) + m(n - (m + 1)/2)\}$ crosspoints and edges.
2.3 SPARSE CROSSBAR CONCENTRATORS

In the previous section, we showed how to construct strong concentrators using graphs with constant fanin and fanout. A particular family of concentrators, called sparse crossbar concentrators, arises when the fanin and fanout of all switching vertices are fixed to 1, and the paths between inputs and outputs are forced to have unit depth. In this main section, we examine two key questions on sparse crossbar concentrators. The first one deals with the crosspoint complexity of a full capacity sparse crossbar concentrator. That is, for any two positive integers, \( n \) and \( m \), how many crosspoints are needed to construct a sparse crossbar \((n,m)\)-concentrator? We will prove that every sparse crossbar \((n,m)\)-concentrator must have \((n - m + 1)m\) crosspoints. This is clearly much stronger a lower bound than the one we were able to derive in Section 2.2.1. The second question is whether or not one can construct a full capacity sparse crossbar \((n,m)\)-concentrator whose crosspoint complexity matches this lower bound. We will give three such constructions, where the first construction is derived from binomial subsets of an \( n \)-set, and the other two are inspired by the lower bound expression on the number of crosspoints.

Before we proceed any further, let us clarify a few points about sparse crossbar concentrators. Within the context of a graph model, a sparse crossbar concentrator can be viewed as a directed tripartite graph (see Figure 2.7(a)) with three disjoint sets of vertices: a set of inputs \( A \), a set of outputs \( B \), and a set of switching vertices \( S \). The edges run between the inputs in \( A \) and the switching vertices in \( S \), and the switching vertices in \( S \) and outputs in \( B \). The fanin and fanout of each vertex in \( S \) are 1, whereas the fanin (fanout) of each output in \( B \) (input in \( A \)) is bounded by the number of inputs (outputs). The set of all output vertices to which an input vertex is connected by a switching vertex is called the neighbor set of that input.

The tripartite graph model does give a complete account of the behavior of a sparse crossbar concentrator, but it is redundant. We can eliminate the vertices in \( S \). This leads to a bipartite graph represen-
Figure 2.7: The tripartite (part (a)), bipartite (part (b)), and crossbar (part (c)) representations of a (4,3)-strong concentrator.
SPARSE CROSSBAR CONCENTRATORS

tation where the role of the switching vertices have been delegated to the input and output vertices as shown in Figure 2.7(b). A sparse crossbar concentrator can also be viewed as a matrix of crosspoints with \( m \) rows and \( n \) columns where a crosspoint exists between column \( i \) and row \( j \) if there exists an edge between input \( i \) and output \( j \) in its bipartite graph representation. This third representation is illustrated in Figure 2.7(c). The squares in the crossbar representation denote the crosspoints; the shaded ones indicating which inputs and outputs are matched as highlighted in the other two representations as well.

We note that each switching vertex in the tripartite representation of a sparse crossbar has fanin = fanout = 1. This implies that, when viewed as a tripartite graph, a sparse crossbar concentrator encompasses as many crosspoints and twice as many edges as switching vertices. However, when viewed as a bipartite graph, edges correspond to crosspoints, and thus, in this case, a sparse crossbar has as many crosspoints and edges as switching vertices. The diameter of a sparse crossbar is 1 in both representations, while its depth is given by \( \log f_i + \log f_o \) where \( f_i \) and \( f_o \) are its fanout and fanin, respectively.

In what follows we will view sparse crossbars strictly as bipartite graphs in which case crosspoints become synonymous with edges.

### 2.3.1 Nakamura-Masson Lower Bound

We start out with a lower bound on the crosspoint complexity of full capacity sparse crossbar concentrators.

**Theorem 2.4** Every sparse crossbar \((n, m)\)-concentrator must contain at least \((n - m + 1)m\) crosspoints.

**Proof:** Let \( G \) be a sparse crossbar \((n, m)\)-concentrator. We will prove the statement by showing that each of \( m \) outputs must be connected to at least \( n - m + 1 \) inputs. Suppose an output of \( G \) is connected to at most \( n - m \) inputs. Then, some \( m \) or more inputs of \( G \) must be connected to only \( m - 1 \) of its \( m \) outputs (See Figure 2.8). But this contradicts the statement that \( G \) is an \((n, m)\)-concentrator, and hence the theorem follows. ||
The sparse crossbar versions of the concentrators described in the previous section obviously use more crosspoints than the lower bound suggested by this theorem. In particular, it is easily established that a sparse crossbar realization of the pruned strong \((n;m)\) concentrator needs \(nm - m^2/2 + m/2\) crosspoints; \(m^2/2 - m/2\) more crosspoints than the lower bound would imply. Of course, this is a strong \((n;m)\)-concentrator, and it is not immediately clear that the lower bound of Nakamura and Masson is tight for such concentrators. For ordinary concentrators, however, Masson constructed a family of \((n;m)\)-concentrators, called binomial networks, whose crosspoint complexity is tight with respect to this lower bound when \(m = \Theta(\sqrt{n})\).

### 2.3.2 Binomial Concentrators

A binomial \(\binom{m}{v}\)-network is a sparse crossbar with \(n = \binom{m}{v}\) inputs and \(m\) outputs, where \(v \leq m\) is a positive integer, such that each input is connected to a distinct \(v\)-subset of outputs. Let \(A = \{a_1, a_2, \ldots, a_n\}\), and \(B = \{b_1, b_2, \ldots, b_m\}\) denote the set of inputs and set of outputs of such a network, respectively. This construction is illustrated in Figure 2.9 for \(n = 6, m = 4, v = 2\). The six 2-subsets of \(B = \{b_1, b_2, b_3, b_4\}\), i.e., \(\{b_1, b_2\}, \{b_1, b_3\}, \{b_1, b_4\}, \{b_2, b_3\}, \{b_2, b_4\}, \{b_3, b_4\}\) are assigned to columns (inputs), \(a_1, a_2, \ldots, a_6\), in that order. It is easily shown by inspection that this sparse crossbar is a \((6, 4)\)-concentrator. Does this fact generally hold? The answer is yes, and its proof is based on Hall’s Theorem.

**Theorem 2.5** *The \(\binom{v+2}{v}\)-network, \(v \geq 2\) is a sparse crossbar concentrator with \(v\binom{v+2}{v}\) crosspoints, fanout = \(v\), and fanin = \(\binom{v+1}{v-1}\).*

**Proof:** We must show that every \(v + 2\) inputs have a matching set. Let \(x_1, x_2, \ldots, x_{v+2}\) be any \(v + 2\) inputs. By Hall’s Theorem, it suffices to show that the union of the neighbor sets of any \(\alpha \leq v + 2\) of \(x_1, x_2, \ldots, x_{v+2}\) contains at least \(\alpha\) outputs. We consider three cases, (a) \(1 \leq \alpha \leq v\), (b) \(\alpha = v + 1\), and (c) \(\alpha = v + 2\). If (a) holds then since each input is connected to \(v\) outputs the statement obviously holds. If (b) holds, then since the neighbor sets of no two inputs
Figure 2.8: Illustration of the argument in Theorem 2.4. The first output is connected to the first $n - m$ inputs, forcing the last $m$ inputs to have at most $m - 1$ outputs as neighbors.

Figure 2.9: A binomial (6,4)-concentrator.
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... coincide, their union should contain at least $v + 1$ outputs so that the statement holds in this case as well. Finally, if (c) holds, then the union of the neighbor sets of all $v + 2$ inputs must contain at least $v + 2$ outputs, since by the construction of the network, with fewer than $v + 2$ outputs, we can fix at most $\binom{v+1}{v} = v + 1$ inputs, not enough to have $v + 2$ distinct inputs. Hence, by Hall’s Theorem, any $v + 2$ inputs has a matching, and therefore, $\binom{v+2}{v}$-network is a full capacity concentrator. That it has $v \binom{v+2}{v}$ crosspoints easily follows by a direct count of the crosspoints. Its fanout is obviously $v$ by its construction. Also, since each output is incident exactly once at an input together with each $(v - 1)$-subset of the remaining $v + 1$ outputs, its fanin is $\binom{v+1}{v-1}$. ||

As we forecast earlier, the number of crosspoints in this concentrator design matches the lower bound of Theorem 2.4 since

$$\left( \frac{v + 2}{v} \right) - (v + 2) + 1 \cdot (v + 2) = v \binom{v+2}{v}.$$ 

This implies the following result in a redundant way (the redundancy will momentarily be apparent).

**Corollary 2.6** The binomial network yields an $(n, \Theta(\sqrt{n}))$-concentrator with $\Theta(n^{1.5})$ crosspoints, fanout $= \Theta(\sqrt{n})$, and fanin $= \Theta(n)$.

**Proof:** Let $n$ be an integer $\geq 3$. Elementary calculus shows that an integer

$$v \in \left[\left( -3 + \sqrt{1 + 8n} \right) / 2, \left( -1 + \sqrt{1 + 8n} \right) / 2 \right]$$

exists such that

$$\left( \frac{(v - 1) + 2}{v - 1} \right) \leq n \leq \left( \frac{v + 2}{v} \right).$$

By Theorem 2.5, we can construct a sparse crossbar concentrator $G$ with $\binom{v+2}{v}$ inputs and $v+2$ outputs. Thus, we can obtain an $(n, v+2)$-concentrator, or, an $(n, \Theta(\sqrt{n}))$-concentrator by discarding $\binom{v+2}{v} - n$ of the inputs of $G$. By Theorem 2.5, and Eqn. 2.9, the number of...
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crosspoints \( \kappa(n, v + 2) \) of this \((n, \Theta(\sqrt{n}))\)-concentrator satisfies the inequality

\[
(v-1)\binom{v+2}{v} \leq \kappa(n, v + 2) \leq v\binom{v+2}{v}, \tag{2.10}
\]

or,

\[
\frac{(v-1)v(v+1)}{2} \leq \kappa(n, v + 2) \leq \frac{v(v + 1)(v + 2)}{2}. \tag{2.11}
\]

Combining this with Eqn. 2.8 shows that, for \( n >> 1 \), there exist positive constants \( c_1, c_2 \) such that \( c_1n^{3/2} \leq \kappa(n, \Theta(\sqrt{n})) \leq c_2n^{3/2} \). Hence the binomial network yields an \((n, \Theta(\sqrt{n}))\)-concentrator with \( \Theta(n^{3/2}) \) crosspoints. The expressions for the fanin and fanout also follow from Theorem 2.5 and Eqn. 2.9, and the depth expression follows from the expressions for fanin and fanout, and Eqn. 2.8.

As we mentioned, this result is not unexpected since an \((n, \Theta(\sqrt{n}))\)-concentrator (even a strong one for that matter!) can be constructed with \( \Theta(n^{3/2}) \) crosspoints most directly by connecting each of its \( n \) inputs to all of its \( \sqrt{n} \) outputs. However, for any integer \( v \geq 2 \), and \( n = \binom{v+2}{v} \), Masson’s binomial network not only has \( \Theta(n^{3/2}) \) crosspoints, but also provides a \((\binom{v+2}{v}, v + 2)\)-concentrator with the minimum number of crosspoints possible.

### 2.3.3 Fat-and-Slim Crossbars

At this point, a natural question to ask is whether or not one can construct \((n, m)\)-concentrators with as few crosspoints as suggested by the lower bound given in Theorem 2.4 for any \( m \leq n \). The answer is yes and it is captured in the following construction.

**Definition 2.4** Let \( \mathcal{G} = (I, O, E) \) be a sparse crossbar with \( n \) inputs and \( m \) outputs. Suppose that \( I \) is partitioned into two sets \( A_1 \) and \( A_2 \), where \( |A_1| = n - m \) and \( |A_2| = m \). \( \mathcal{G} \) is called an \((n, m)\)-fat-and-slim crossbar if each of the \( n - m \) inputs in \( A_1 \) is connected to all the \( m \) outputs, and if each of the \( m \) inputs in \( A_2 \) is connected to a single but distinct output.
Theorem 2.7 (Oruç and Huang) Every \((n, m)\)-fat-and-slim crossbar is a sparse crossbar concentrator with a minimum number of crosspoints for any positive integers \(m \leq n\).

Proof: The crosspoint complexity of any fat-and-slim crossbar clearly matches the lower bound in Theorem 2.4. Let \(G = (I, O, E)\) be an \((n, m)\)-fat-and-slim crossbar, \(x_1, x_2, \ldots, x_m\) be any \(m\) inputs of \(G\), and let \(Y_i\) be the neighbor set of input \(x_i, 1 \leq i \leq m\). By the construction of \(G\), it is obvious that if at least one of \(x_1, x_2, \ldots, x_m\) belongs to \(A_1\) then the union of any \(\alpha\) of \(Y_1, Y_2, \ldots, Y_m\) contains at least \(m \geq \alpha\) outputs. On the other hand, if all of \(x_1, x_2, \ldots, x_m\) belong to \(A_2\) then the union of any \(\alpha\) of \(Y_1, Y_2, \ldots, Y_m\) contains exactly \(\alpha\) outputs. Therefore, by Hall’s theorem, \(G\) is an \((n, m)\)-concentrator, and the statement follows.

This construction is illustrated in Figure 2.10 for \(n = 12\) and \(m = 4\). Letting \(m = n^\epsilon\) in the crosspoint complexity expression of a fat-and-slim crossbar shows that, for any integral \(n^\epsilon\), where \(0 < \epsilon < 1\), a sparse crossbar \((n, n^\epsilon)\)-concentrator can be constructed with \(n^{1+\epsilon} - n^{2\epsilon} + n^\epsilon\) crosspoints, and by Theorem 2.4, this is the least crosspoint count possible for such a network.
2.3.4 Balanced Fat-and-Slim Crossbars

The fat-and-slim crossbar resolves the question of constructing a sparse crossbar \((n, m)\)-concentrator with the minimum number of crosspoints for any \(m \leq n\), but with an irregular fanout. It is reasonable to ask if a sparse crossbar \((n, m)\)-concentrator can be constructed with \((n - m + 1)m\) crosspoints so that all inputs have nearly the same fanout and all outputs have nearly the same fan in. The following theorem settles the question in the affirmative when \(n = 2m\).

**Theorem 2.8 (Oruç and Huang)** Let \(G = (I, O, E)\) be a sparse crossbar with \(2m\) inputs and \(m\) outputs. Let \(O = \{1, 2, 3, \ldots, m\}\) and \(I = \{m + 1, m + 2, \ldots, 2m\}\). Suppose each input in \(I_1\) is connected to all the odd outputs, and also input \(2i\) is connected to output \(2i, 1 \leq i \leq \lfloor m/2 \rfloor\). Likewise, suppose each input in \(I_2\) is connected to the even outputs, and also input \(2i + 1\) is connected to output \(2(i - \lfloor m/2 \rfloor) + 1, \lfloor m/2 \rfloor \leq i \leq m - 1\) (See Figure 2.11). \(G\) is a \((2m, m)\)-concentrator with a minimum number of crosspoints, fanout = \(\lceil m/2 \rceil + 1\) and fanin = \(m + 1\).

**Proof:** Let \(X\) be an arbitrary \(r\)-subset of inputs, \(1 \leq r \leq m\), and let \(X_1 = X \cap I_1 = \{x_1, x_2, \ldots, x_p\}\) and \(X_2 = X \cap I_2 = \{x'_1, x'_2, \ldots, x'_q\}\), where \(r = p + q\). Let \(Y_i\) be the neighbor set of input \(x_i, 1 \leq i \leq p\), and let \(Y'_i\) be the neighbor set of input \(x'_i, 1 \leq i \leq q\). It is easy to see that, for any \(p \geq 1\), \(Y_1 \cup Y_2 \cup \ldots \cup Y_p\) contains at least \(\lfloor m/2 \rfloor + \epsilon_1\) outputs, where the first term in the sum accounts for the odd numbered neighbors of the inputs in \(X \cap I_1\), and \(\epsilon_1 \geq 0\) accounts for the even numbered neighbors of the same inputs. Likewise, for any \(q \geq 1\), \(Y'_1 \cup Y'_2 \cup \ldots \cup Y'_q\) contains at least \(\lfloor m/2 \rfloor + \epsilon_2\) outputs, where the first term in the sum accounts for the even numbered neighbors of the inputs in \(X \cap I_2\), and \(\epsilon_2 \geq 0\) accounts for the odd numbered neighbors of the same inputs. Now, let

\[
Y = Y_1 \cup Y_2 \cup \ldots \cup Y_p
\]

and

\[
Y' = Y'_1 \cup Y'_2 \cup \ldots \cup Y'_q.
\]
Figure 2.11: A \((2m, m)\)-concentrator with a minimum number of crosspoints and uniform fanin and fanout \((m = 6)\).

Then the number of neighbors of the inputs in \(X\) is given by

\[ |Y| + |Y'| - |Y \cap Y'|. \]

Furthermore, the indices of the outputs in \(Y\) and \(Y'\) show that the intersection of the two sets contains no more than

\[ \text{Min}\{\epsilon_1, m/2\} + \text{Min}\{\epsilon_2, m/2\} = \epsilon_1 + \epsilon_2 \]

outputs. Therefore, the inputs in \(X\) must have at least

\[ \lceil m/2 \rceil + \epsilon_1 + \lceil m/2 \rceil + \epsilon_2 - (\epsilon_1 + \epsilon_2) = m \]

neighbors for any \(p, q \geq 1\) and \(1 \leq r = p + q \leq m\). Hence, by Hall’s theorem, \(G\) is an \((n, m)\)-concentrator. As for its crosspoint complexity, it is seen that every output is connected to \(m+1\) outputs so that \(G\) contains \((m + 1)m = (2m - m + 1)m\) crosspoints and this matches the lower bound of Theorem 2 for \(n = 2m\). The fanin and fanout expressions are obvious. ||

It is possible to extend this sparse crossbar concentrator construction to sparse crossbars with other input and output profiles. The
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main idea is to distribute the crosspoints in the sparse portion of a fat-and-slim crossbar more evenly over the inputs. The following construction offers one such extension.

**Theorem 2.9 (Guo-Liang-Oruç)** Let $G = (I, O, E)$ be a fat-and-slim crossbar with $km$ inputs and $m$ outputs, where $k$ is a divisor of $m$. Let

$$A_G = \begin{bmatrix}
A_{1,1} & A_{1,2} & \ldots & A_{1,k} \\
A_{2,1} & A_{2,2} & \ldots & A_{2,k} \\
\vdots & & & \\
A_{k,1} & A_{k,2} & \ldots & A_{k,k}
\end{bmatrix}$$

be the adjacency matrix of $G$, partitioned into $k^2$ matrices, where $A_{i,j}$ is an $(m/k) \times m$ matrix, $1 \leq i, j \leq k$. Let $G'$ be a sparse crossbar whose adjacency matrix is obtained from $A_G$ by swapping $A_{i,i}$ with $A_{i,k}, 1 \leq i \leq k-1$. Then $G'$ is a $(km,m)$-concentrator with a minimum number of crosspoints, fanout $= m(k-1)/k + 1$ and fanin $= n - m + 1$.

We leave the proof of this theorem to the reader, and illustrate the construction for $n = 16$ and $m = k = 4$. In this case, we start out with a $(16,4)$-fat-and-slim crossbar, and swap $A_{1,1}$ with $A_{1,4}$, $A_{2,2}$ with $A_{2,4}$, and $A_{3,3}$ with $A_{3,4}$ in the adjacency matrix of $G$. The resulting adjacency matrix then gives the sparse crossbar $(16,4)$-concentrator in Figure 2.12. It should also be noted that the condition that $k|m$ can be relaxed with a slight effect on the regularity of the construction (See Problem 2.19.)

**2.3.5 Banded Sparse Crossbars**

Balanced fat-and-slim crossbars are just an example of how to transform a known concentrator into another one without altering its crosspoint complexity. In this section, we introduce banded spare crossbars as another example.

**Definition 2.5** An $(n,m)$-sparse crossbar $G$ is called (column) banded if its adjacency matrix $A_G = [a_{i,j}]$ is given by

$$a_{i,j} = \begin{cases}
1 & \text{if } i \leq j \leq i + n - m \\
0 & \text{if } j < i \text{ or } j > i + n - m
\end{cases} \quad (2.12)$$
Figure 2.12: A balanced \((km, m)\) fat-and-slim crossbar with a minimum number of crosspoints and uniform fanin and fanout \((k = m = 4)\).

for \(i = 1, 2, \ldots, m\).

Figure 2.13 depicts a \((9, 4)\)-banded sparse crossbar. It is seen that the crosspoints of a banded sparse crossbar are distributed more evenly over its columns than those in a fat-and-slim sparse crossbar, but not as evenly as in a balanced fat-and-slim crossbar. We also note that the \((n, m)\)-banded sparse crossbar has a fixed fanin of \(n - m + 1\) as would be the case for an \((n, m)\)-fat-and-slim crossbar.

We will use the following transformation theorem to prove that every banded sparse crossbar is a full capacity concentrator with a minimum number of crosspoints.

**Theorem 2.10 (Guo-Oruç)** Let \(X\) and \(Y\) be any two columns of an \(n \times m\) binary matrix \(A\). Suppose \(X\) covers \(Y\), and let \(a_{i_1,x}, \ldots, a_{i_r,x}\) be a set of \(r\) “1”s in \(X\). Let \(B\) be a matrix obtained from \(A\) by exchanging \(a_{i_l,x}\) with \(a_{i_l,y}\), \(1 \leq l \leq r\). If \(A\) does not have a \((m - k + 1) \times k\) zero submatrix, for any \(k, 1 \leq k \leq m\), then neither does \(B\).

**Proof:** Suppose \(A\) does not have a \((m - k + 1) \times k\) matrix but \(B\) does. Since the zero rows in \(X\) and \(Y\) are preserved by the exchange operation, if the zero submatrix does not contain either \(X\) or \(Y\), or it contains both \(X\) and \(Y\), \(A\) must also contain the same zero submatrix, a contradiction. If the zero submatrix in \(B\) contains \(Y\) but not \(X\), then since \(Y\) in \(B\) covers \(Y\) in \(A\), \(A\) must also contain the
same zero submatrix. Finally, if the zero submatrix in $B$ contains $X$ but not $Y$, then since $X$ in $B$ covers $Y$ in $A$, exchanging $X$ with $Y$ in $A$ shows that $A$ also contains a zero submatrix, again a contradiction, and the statement follows. \[\square\]

The following is a direct corollary of Hall’s Theorem and the previous statement.

**Corollary 2.11** Let $G$ be a sparse crossbar concentrator with capacity $c \leq m$ and two inputs $x$ and $y$ where $x$ covers $y$. If a subset of rows of $x$ are exchanged with the corresponding rows in $y$ then the resulting sparse crossbar is also a concentrator with capacity $c$.

**Theorem 2.12** The banded $(n, m)$-sparse crossbar is a full capacity concentrator with a minimum crosspoint complexity for all $n \geq m$.

**Proof:** We prove the statement inductively by showing that a banded $(n, m)$-sparse crossbar concentrator can be obtained by applying the column exchange operation described in the previous theorem to a fat-and-slim crossbar. Suppose $G$ is an $(n, m)$-fat-and-slim crossbar, where $n = m + 1$. Clearly, column 1 of $G$ covers column $m$ of $G$ so that the $m$th row of column 1 can be exchanged with the $m$th row of column $m$. Furthermore, after the exchange, column 1 still covers
Figure 2.14: The steps showing how a $(5, 4)$-fat-and-slim concentrator is transformed into a $(5, 4)$-banded concentrator by column-exchange operations.
column $m-1$, so that the $(m-1)$th row of column 1 can be exchanged with $(m-1)$th row of column $m-1$, and this process can be repeated until the 2nd row of column 1 is exchanged with the 2nd row of column 2. The resulting matrix is obviously banded and by Corollary 3.1, it yields a concentrator. (See Figure 2.14 for an illustration for $n = 5, m = 4$). Now suppose that an $(m + k, m)$-fat-and-slim sparse crossbar can be transformed into a banded $(m + k, m)$-sparse crossbar concentrator. Let $G$ be an $(m + k + 1, m)$-fat-and-slim crossbar. $G$ can be expressed as a direct sum of two sparse crossbars $G_1$ and $G_2$, where $G_1$ is a $(1, m)$ full crossbar, and $G_2$ is an $(m + k, m)$-fat-and-slim sparse crossbar. By the induction assumption, $G_2$ can be transformed into a banded $(m + k, m)$-sparse crossbar concentrator, $G_2'$. Moreover, the crosspoints in rows $2, 3, \ldots, m$ of $G_1$ can be distributed into the corresponding rows of the first $m$ columns in $G_2'$ by iteratively applying the previous corollary to obtain a banded $(m + k + 1, m)$-sparse crossbar concentrator (See Figure 2.15). The statement follows by induction.

We will now make use of a banded sparse crossbar concentrator to obtain a regular $(n, m)$-sparse crossbar concentrator with minimum crosspoint complexity for any positive integers $m, n \geq m$. Let

$$\alpha = \begin{cases} 
\left\lfloor \frac{(n-m+1)m}{n} \right\rfloor & \text{if } n < \frac{3m}{2} \\
\left\lceil \frac{(n-m+1)m}{n} \right\rceil & \text{if } \frac{(n-m+1)m}{n} - \left\lfloor \frac{(n-m+1)m}{n} \right\rfloor < 0.5, \text{ and } n \geq \frac{3m}{2} \\
\left\lceil \frac{(n-m+1)m}{n} \right\rceil & \text{if } \frac{(n-m+1)m}{n} - \left\lfloor \frac{(n-m+1)m}{n} \right\rfloor \geq 0.5, \text{ and } n \geq \frac{3m}{2}
\end{cases}$$

$$\beta = n - m - \alpha + 1.$$ (2.13)

Let $G_1$ be a $(\beta, m)$-full crossbar and $G_2$ be an $(m + \alpha - 1, m)$-banded crossbar concentrator. Let $G = G_1 + G_2$. It can be verified that $G$ is an $(n, m)$-sparse crossbar concentrator whose adjacency matrix $A_G = [a_{i,j}]$ is given by

$$a_{i,j} = \begin{cases} 
1 & \text{if } i + \beta \leq j \leq i + n - m \\
1 & \text{if } 1 \leq j \leq \beta \\
0 & \text{if } \beta < j < i + \beta \text{ or } j > i + n - m + \beta
\end{cases}$$ (2.14)

for $i = 1, 2, \ldots, m$. 

Figure 2.15: Illustration of the induction step in Theorem 2.12.
$A_G$ can be decomposed as $[J|U|B|L]$, where

- $J$ is a $m \times \beta$ matrix of “1”s,
- $U$ is an $m \times (\alpha - 1)$ upper triangular matrix,
- $B$ is a transposed $m \times (m - \alpha + 1)$-banded matrix, and
- $L$ is a $m \times (\alpha - 1)$ lower triangular matrix.

Figure 2.16 illustrates this decomposition for a concentrator with 11 inputs and 5 outputs, $\alpha = 3$, and $\beta = 4$.

We now establish that the crosspoints of this sparse crossbar construction can be balanced so that every column has either $\alpha \pm 1$ or $\alpha$ crosspoints, and every row has $n - m + 1$ crosspoints.

**Theorem 2.13** Let $A_G = [J|U|B|L]$ as defined above. $A_G$ can be rearranged using the column exchange operation described in Theorem 2.10 to obtain a matrix in which every column has either $\alpha \pm 1$ or $\alpha$ “1”s.

**Proof:** We consider, two cases:

**Case 1** $(4 \leq m \leq 2n/3)$: By the construction of $A_G$, the columns of $B$ already have $\alpha$ ”1”s. So we need only to consider the columns of $A_G/B$. We first note that the total number of “1”s in matrix $A_G/B$ is $(n - m + 1)m - \alpha(n - \beta - 2\alpha + 2)$ and $A_G/B$ has $\beta + 2\alpha - 2$ columns. Recalling the values of $\alpha$ and $\beta$ from Eqn. (2.13), the average number of “1”s in a column of $A_G/B$ is

$$
\frac{(n - m + 1)m - \alpha(n - \beta - 2\alpha + 2)}{\beta + 2\alpha - 2} = \frac{(n - m + 1)m - \alpha n}{\beta + 2\alpha - 2} + \alpha
$$

$$
= \begin{cases} 
\frac{(n - m + 1)m - [(n - m + 1)m/n]}{n - m + [(n - m + 1)m/n]} + \alpha & \text{if } (n - m + 1)m/n - [(n - m + 1)m/n] < 0.5 \\
\frac{(n - m + 1)m - [(n - m + 1)m/n] - 1}{n - m + [(n - m + 1)m/n] - 1} + \alpha & \text{if } (n - m + 1)m/n - [(n - m + 1)m/n] \geq 0.5.
\end{cases}
$$

(2.15)

We will show that the absolute value of the fraction term in each of the expressions is bounded by 1. First, suppose

$$
0 \leq (n - m + 1)m/n - [(n - m + 1)m/n] < 0.5.
$$

(2.16)

It can then be verified by direct calculation that, for $4 \leq n \leq 19$, and $2 \leq m \leq 12$

$$
0 \leq \frac{(n - m + 1)m - [(n - m + 1)m/n]}{n - m + [(n - m + 1)m/n] - 1} < 1.
$$
Figure 2.16: An (11,5)-concentrator, and its constituent networks.
For $n \geq 20$, Eqn. 2.16 implies

$$0 \leq \frac{(n - m + 1)m - [(n - m + 1)m/n]n}{n - m + [(n - m + 1)m/n] - 1} < \frac{0.5n}{n - m + [(n - m + 1)m/n] - 1}. $$

Since the denominator is a decreasing function of $m$ for $m \leq n$, the maximum value of the expression on the right occurs at $m = 2n/3$ so that

$$0 \leq \frac{(n - m + 1)m - [(n - m + 1)m/n]n}{n - m + [(n - m + 1)m/n] - 1} < \frac{0.5n}{n/3 + [(n/3 + 1)/3] - 1}. $$

Furthermore, the envelop of the function on the right is monotone decreasing as $n$ increases, and it is $\leq 0.9$ for $n \geq 20$.

Similarly, it can be verified that if

$$(n - m + 1)m/n - [(n - m + 1)m/n] \geq 0.5, \quad (2.17)$$

then

$$\frac{-0.5n}{n - m + [(n - m + 1)m/n] - 1} \leq \frac{(n - m + 1)m - [(n - m + 1)m/n]n}{n - m + [(n - m + 1)m/n] - 1} \leq 0.$$

The minimum value of the expression on the left occurs at $m = 2n/3$ so that

$$\frac{-0.5n}{n/3 + [(n/3 + 1)/3] - 1} \leq \frac{(n - m + 1)m - [(n - m + 1)m/n]n}{n - m + [(n - m + 1)m/n] - 1} \leq 0.$$

Furthermore, the envelop of the function on the left is increasing with $n$, and it is $> -1$ for all $n \geq 7$. Finally, direct computation shows that

$$-1 \leq \frac{(n - m + 1)m - [(n - m + 1)m/n]n}{n - m + [(n - m + 1)m/n] - 1} \leq 0.$$
It follows that, when \( n \geq \frac{3m}{2} \) the average number of “1”s over the columns of \( A_G \) is in the region \([\alpha-1, \alpha+1]\). Moreover, it is easy to see that the columns of \( U \) and \( L \) can be balanced to have \( \alpha \pm 1 \) crosspoints by moving \( m - \alpha \) crosspoints from each of the columns in \( J \) using Theorem 2.10. This leaves each column in \( J \) with \( \alpha \) crosspoints, and hence, \( A_G \) can be balanced so that each of its columns has \( \alpha \pm 1 \) or \( \alpha \) “1”s.

**Case 2** \((2n/3 < m \leq n - 1)\): In this case, suppose we initially assign \( \alpha \) “1”s to each of the \((n - \beta - 2\alpha + 2)\) columns in \( B \) and \((\beta + 2\alpha - 2)(\alpha + 1)\) “1”s to the columns in \( A_G - B \), where \( \alpha \) as defined in Eqn. 2.13. Then the number of “1”s which are left unassigned is given by

\[
\gamma = (n - m + 1)m - \alpha n - (\beta + 2\alpha - 2). \tag{2.18}
\]

Since \((n - m + 1)m - \alpha n \geq 0\), we must have \( \gamma \geq -(\beta + 2\alpha - 2) \). Thus, if \( \gamma \leq 0 \), then the average deficit of “1”s among the \((\beta + 2\alpha - 2)\) columns cannot exceed 1 so that the average number of “1”s in \( A_G \) must be in the interval \([\alpha, \alpha + 1]\). Therefore, the column-exchange procedure described in Theorem 2.10 can be used to balance the columns in \( A_G / B \) so that each column in \( A_G \) has either \( \alpha \) or \( \alpha + 1 \) “1”s. On the other hand, if \( \gamma > 0 \), then we have a surplus of \( \gamma \) “1”s that must be distributed over the columns of \( A_G \). In this case, we first preserve \( \left\lfloor \frac{\gamma}{m - \lfloor \alpha - 1 \rfloor} \right\rfloor \) columns of “1”s in \( J \), and balance the remaining columns in \( J \) with the columns in \( U \) and \( L \) so that each of the balanced columns has \( \alpha + 1 \) “1”s. Now, the inequality

\[
0 \leq (n - m + 1)m - \alpha n \leq n,
\]

together with Eqn. 2.18 implies \( \gamma \leq n - (\beta + 2\alpha - 2) \), where \( n - (\beta + 2\alpha - 2) \) is the number of columns in \( B \). Therefore, one can distribute the extra \( \gamma \) “1”s from the unbalanced columns in \( J \) to the columns in \( B \) with each having at most one additional ‘1’. This concludes the proof. ||

We thus have the proved the following main result.

---

\( ^3 \) It can be verified that the total number of "1"s to be moved from the \( J \) matrix is equal to the total number of "0"s in \( U \) and \( L \). That is, \((m - \alpha)(n - m - \alpha + 1) = \alpha(\alpha - 1)\).
Corollary 2.14 (Guo-Oruç) For any $n \geq m$, there exists a balanced $(n,m)$-sparse crossbar concentrator with $(n - m + 1)m$ crosspoints, $\text{fanin} = n - m + 1$, and $\text{fanout} = \left\lfloor \frac{(n-m+1)m}{n} \right\rfloor \pm 1$.

Figure 2.17 shows how the crosspoints in the sparse crossbar of Figure 2.16 are balanced by distributing the “1”s in $J$ into $U$ and $L$, when $n < 3m/2$. In this case, $\gamma = -6 < 0$ so that crosspoints are balanced without splitting $J$ into two parts as outlined in the proof. We note, however, that $\gamma$ can indeed be $> 0$ as the reader is asked to verify in Problem 2.33.

2.4 BOUNDED CAPACITY SPARSE CROSSBARS

Much of our focus so far has been on sparse crossbar full capacity concentrators. In this main section, we will consider the problem of constructing bounded capacity concentrators. As in sparse crossbar full capacity concentrators, our main problem is to construct sparse crossbar concentrators within an achievable lower bound of
crosspoint complexity. As we will see, this will prove to be a more difficult problem than that for sparse crossbar full capacity concentrators.

Recall that, in Section 2.1, we defined a c-bounded concentrator, or $(n, m, c)$-concentrator as a directed multipartite graph in which the maximum number of inputs that can be concentrated is bounded by c. It was shown in Section 2.2.1 that any $(n, m, c)$-bounded concentrator requires $\Omega(n)$ crosspoints and edges. Here, we will consider sparse crossbar (bipartite) realizations of bounded concentrators, and derive two tighter lower bounds on the number of crosspoints required by any such concentrator. The first of these bounds is simple to compute, but not always tight. The second derivation provides a tighter bound (even though not always tight either), but requires more computation. We will conclude this main section by presenting some $(n, m, c)$-concentrators whose crosspoint complexities match the lower bounds.

### 2.4.1 A Lower Bound On Crosspoint Complexity

The first lower bound is obtained by extending Theorem 2.4.

**Theorem 2.15** Any sparse crossbar $(n, m, c)$-concentrator requires

$$\left\lfloor \frac{m(n - c + 1)}{m - c + 1} \right\rfloor$$

crosspoints.

**Proof:** Let $G = (I, O, E)$ be a sparse crossbar $(n, m, c)$-concentrator. Then each $(m - c + 1)$-subset of outputs should be connected to at least $n - c + 1$ inputs. Otherwise, some $c$ inputs will only be connected to $c - 1$ outputs, contradicting the fact that $G$ is an $(n, m, c)$-concentrator. Let $P_{m-c+1}(O)$ denote the collection of all $(m - c + 1)$-subsets of $O$, $d_i$ denote the fanin of output $i$, $1 \leq i \leq m$, and $p_i$ denote the number of $(m - c + 1)$-subsets of $O$ that contain output $i$. Since the number of neighbors (inputs) of the outputs in $O$...
an \((m - c + 1)\)-subset of \(O\) cannot be larger than the sum of the crosspoints connected to the outputs in that subset, we have

\[
\sum_{Y \in P_m (O)} \sum_{i \in Y} d_i \geq \binom{m}{m - c + 1} (n - c + 1), \tag{2.20}
\]

or equivalently,

\[
\sum_{i=1}^{m} \rho_i d_i \geq \binom{m}{m - c + 1} (n - c + 1), \tag{2.21}
\]

where the expressions on the left hand side of both inequalities sum the fanins of the outputs in \(O\) over all of its \((m - c + 1)\)-subsets.

Now, let \(\kappa_G(n, m, c)\) denote the number of crosspoints in \(G\). Noting that \(\rho_i = \binom{m-1}{m-c}\) and \(\kappa_G(n, m, c) = \sum_{i=1}^{m} d_i\), Eqn. 2.21 gives

\[
\binom{m-1}{m-c} \kappa_G(n, m, c) \geq \binom{m}{m - c + 1} (n - c + 1), \tag{2.22}
\]

or,

\[
\kappa_G(n, m, c) \geq \left\lfloor \frac{\binom{m}{m-c+1}(n - c + 1)}{\binom{m-1}{m-c}} \right\rfloor \tag{2.23}
\]

which reduces to \(\kappa_G(n, m, c) \geq \left\lfloor \frac{m(n - c + 1)}{m-c+1} \right\rfloor\) when simplified.

An example may help clarify the derivation of this lower bound. Let \(n = 7, m = 6,\) and \(c = 2\). The lower bound is based on the fact that every \(m - c + 1 = 5\) outputs should be connected to at least \(n - c + 1 = 6\) inputs. If any 5 outputs are connected to fewer than 6 inputs then some two inputs can only be connected to at most 1 output, disqualifying the sparse crossbar in question from being a \((7, 6, 2)\)-concentrator. With this requirement in place, Eqn. 2.20 states that the sum of the fanins of the outputs in any sparse \((7, 6, 2)\)-concentrator over all 5-subsets of its outputs must be \(\geq \binom{6}{6-2+1}(7 - 2 + 1) = 36\). The sum, in this case, is given by

\[
\begin{align*}
&d_1 + d_2 + d_3 + d_4 + d_5 \\
+ &d_1 + d_2 + d_3 + d_4 + d_6
\end{align*}
\]
This simplifies to $5(d_1 + d_2 + d_3 + d_4 + d_5 + d_6)$ (the sum on the left hand side of Eqn. 2.21) where 5 is the value of $\rho_i, 1 \leq i \leq 6$, i.e., the number of 5-subsets of outputs that contain each of the six outputs. So, the lower bound in this case is $\sum_{i=1}^{6} \binom{6}{i} (7-2i+1) / \binom{6}{i-2} = \lceil 36/5 \rceil = 8$.

We note that this lower bound reduces to the lower bound given in Theorem 2.4 when $c = m$, and to the lower bound given in Eqn. 2.5 when $c = 1$. For other values of $c$, the bound is not tight. To see this, consider the crossbar network in Figure 2.18. By inspecting all pairs of inputs, it is easily shown that this network is an $(8, 4, 2)$-concentrator. For $n = 8, m = 4, c = 2$, the above lower bound gives $\lceil \binom{4}{2} (7)/\binom{3}{2} \rceil = 10$ crosspoints. However, examination of the $(8, 4, 2)$-concentrator in Figure 2.18 reveals that at least twelve crosspoints are needed to form such a concentrator. This can be justified as follows: First, a minimum of four crosspoints are required between the first four inputs and the four outputs and likewise a minimum of four crosspoints are required between the last four inputs and the
four outputs. These are indicated by the hollow rectangles in the figure. The two crosspoints in each row can be moved to any two columns as long as each input is tied to a crosspoint. However, it is easy to see that no matter how these eight crosspoints are distributed there will exist four pairs of inputs without a matching set. In the figure, with these eight crosspoints in place, none of the pairs \{a_1, a_5\}, \{a_2, a_6\}, \{a_3, a_7\}, and \{a_4, a_8\} have a matching set. Since there is no overlap among these pairs of inputs, each pair needs one more crosspoint (shaded rectangles) in order to have a matching set of outputs. Thus, a sparse crossbar \((8, 4, 2)\)-concentrator requires at least twelve crosspoints, two more than the above lower bound would indicate.

2.4.2 A Tighter Lower Bound

Masson and Nakamura obtained another lower bound which is, in general, tighter than the lower bound of Theorem 2.15, but requires more effort to compute. This bound is derived as follows. As before, let \(G = (I, O, E)\) be a sparse crossbar and let \(M_i\) be a subset of \(O\). The isolation set, \(\sigma(M_i)\), of \(M_i\) is the subset of inputs in \(B\) that are connected only to outputs in \(M_i\) by a crosspoint. If input \(j \in \sigma(M_i)\) then \(M_i\) is said to isolate that input. For example, if \(M_i = \{b_1, b_2\}\) in the network of Figure 2.18 then \(\sigma(M_i) = \{a_1, a_2, a_5\}\), and \(M_i\) isolates each of \(a_1\), \(a_2\), and \(a_5\).

The lower bound hinges on the following lemma:

**Lemma 2.16 (Nakamura-Masson)** Let \(G\) be any \((n, m, c)\)-concentrator, and \(M_i\) be any \(c\)-subset of outputs of \(G\). Let \(t_i\) denote the number of crosspoints that connect the outputs in \(M_i\) to the inputs in \(\sigma(M_i)\). Then,

\[
t_i \geq (|\sigma(M_i)| - c + 1)c.
\]

**Proof:** We proceed as in Theorem 2.4 with \(M_i\) replacing the set of all outputs of \(G\) and \(\sigma(M_i)\) replacing the set of all of its inputs. ||
Now, summing both sides of the above inequality over all \((m \choose c)\) \(c\)-subsets of \(O\) gives

\[
\sum_{i=1}^{n} t_i \geq \left\{ \sum_{i=1}^{n} |\sigma(M_i)| - \left( \begin{array}{c} m \\ c \end{array} \right) (c - 1) \right\} c. \tag{2.25}
\]

Let \(W(j)\) denote the number of inputs of \(G\) that are connected to exactly \(j\) outputs. Noting that the summation on the left hand side of the above inequality counts the number of crosspoints tied to the inputs in the isolation sets of all \(c\)-subsets of \(O\), and also that exactly \(\left( \begin{array}{c} m-j \\ c-j \end{array} \right)\) of the \(c\)-subsets isolate each input that is connected to exactly \(j\) outputs, we must have\(^5\)

\[
\sum_{i=1}^{n} t_i = \sum_{j=1}^{m} \left( \begin{array}{c} m-j \\ c-j \end{array} \right) jW(j). \tag{2.26}
\]

On the other hand, the summation on the right hand side of the same inequality counts the number of inputs in the isolation sets of all \(c\)-subsets of \(Y\), and hence

\[
\sum_{i=1}^{n} |\sigma(M_i)| = \sum_{j=1}^{m} \left( \begin{array}{c} m-j \\ c-j \end{array} \right) W(j). \tag{2.27}
\]

Combining Eqns. 2.25,2.26 and 2.27 together yields

\[
\sum_{j=1}^{m} \left( \begin{array}{c} m-j \\ c-j \end{array} \right) jW(j) \geq \left\{ \sum_{j=1}^{m} \left( \begin{array}{c} m-j \\ c-j \end{array} \right) W(j) - \left( \begin{array}{c} m \\ c \end{array} \right) (c - 1) \right\} c. \tag{2.28}
\]

Rearranging the terms, we get

\[
\left( \begin{array}{c} m \\ c \end{array} \right) (c - 1)c \geq \sum_{j=1}^{m} \left( \begin{array}{c} m-j \\ c-j \end{array} \right) (c - j)W(j) \tag{2.29}
\]

\(^5\) Note that the summation index \(j\) ought to run from 1 to \(c\). However, since \(\left( \begin{array}{c} m-j \\ c-j \end{array} \right) = 0\) for \(c+1 \leq j \leq m\), \(\sum_{j=1}^{m} \left( \begin{array}{c} m-j \\ c-j \end{array} \right) jW(j) = \sum_{j=1}^{m} \left( \begin{array}{c} m-j \\ c-j \end{array} \right) jW(j)\).
and upon dividing each side by $n \binom{m}{c}$, we find

$$(c - 1)c/n \geq \frac{c!}{m!} \sum_{j=1}^{m} \frac{(m - j)!}{(c - j)!} (c - j)W(j)/n.$$ (2.30)

Let

$$f(j) = (m - j)!/(c - j)!$$ (2.31)

Eqn. 2.30 becomes

$$(c - 1)c/n \geq \frac{c!}{m!} \sum_{j=1}^{m} f(j)W(j)/n.$$ (2.32)

It can be verified that $df(j)/dj < 0$, and $d^2 f(j)/dj^2 > 0$ (for example, by using induction on $j$) in the interval $[1, c]$ so that $f(j)$ is a convex decreasing function for all $j, 1 \leq j \leq c$. Hence, by Jensen’s inequality\(^6\),

$$\sum_{j=1}^{m} f(j)W(j)/n \geq f \left( \sum_{j=1}^{m} jW(j)/n \right).$$ (2.33)

Combining Eqns. 2.32 and 2.33,

$$f \left( \sum_{j=1}^{m} jW(j)/n \right) \leq \frac{(c - 1)c(m!)}{n(c!)}. \quad (2.34)$$

Now, we note that, for any $(n, m, c)$-concentrator, $\sum_{j=1}^{m} W(j) = n$, and hence $0 \leq W(j)/n \leq 1, 1 \leq j \leq m$, and $\sum_{j=1}^{m} W(j)/n = 1$. Let

$$x_m = \sum_{j=1}^{m} (jW(j))/n.$$ 

By definition, $x_m$ is the mean of the fanouts of the inputs of $G$, and is called the incident mean of $G$. Now, suppose $x_s$ is the solution

\(^6\) Given any function $f$ which is convex in an interval $[a, b]$, and positive fractions $\alpha_i, 1 \leq i \leq m$, where $\alpha_1 + \alpha_2 + \ldots + \alpha_m = 1$, Jensen’s inequality states that $\sum_i^m \alpha_i f(x_i) \geq f(\sum_{i=1}^{m} \alpha_i x_i)$ holds for any $x_1, x_2, \ldots, x_m, a \leq x_1 \leq x_2 \leq \ldots \leq x_m \leq b$.\]
of \( f(x_s) = (c-1)c(m!) \). Since \( f \) is monotone decreasing and \( f(x_m) \leq (c-1)c(m!) / n(c!) \), we must have \( x_m \geq x_s \). Furthermore, since \( \sum_{j=1}^{m} jW(j) = nx_m \geq nx_s \) gives the number of crosspoints in \( G \), we have the following theorem.

**Theorem 2.17 (Nakamura-Masson)** Any sparse crossbar \((n, m, c)\)-concentrator must have at least \( nx_s \) crosspoints where \( x_s \) satisfies

\[
\frac{(m - x_s)!}{(c - x_s - 1)!} = \frac{(c - 1)c(m!)}{n(c!)}
\]

or, equivalently,

\[
\frac{\binom{c}{x_s}}{\binom{m}{x_s}} n(c - x_s) - c^2 + c = 0.
\]

Let us clarify the steps of the preceding proof with an example. Consider the sparse crossbar in Figure 2.18, let \( c = 3 \). Then, the 3-subsets of outputs of \( G \) and their isolation sets are

\[
\begin{align*}
M_1 &= \{b_1, b_2, b_3\}, & \sigma(M_1) &= \{a_1, a_2, a_3, a_5, a_6\} \\
M_2 &= \{b_1, b_2, b_4\}, & \sigma(M_2) &= \{a_1, a_2, a_4, a_5, a_8\} \\
M_3 &= \{b_1, b_3, b_4\}, & \sigma(M_3) &= \{a_1, a_3, a_4, a_7, a_8\} \\
M_4 &= \{b_2, b_3, b_4\}, & \sigma(M_4) &= \{a_2, a_3, a_4, a_6, a_7\}
\end{align*}
\]

Since each of \( a_1, a_2, a_3 \) and \( a_4 \) is connected to one output, and each of the remaining inputs is connected to two outputs, it is clear that \( t_i = 7, 1 \leq i \leq 4 \), and hence the inequality in Eqn. 2.24 does not hold, implying that the sparse crossbar in Figure 2.18 cannot have a capacity of 3. To understand why the identity in Eqn 2.26 holds, we just note that both sums in the identity count the number of crosspoints incident upon all the inputs in the multiset union of \( \sigma(M_i), 1 \leq i \leq 4 \) in two different ways. The sum on the left computes this number by totaling the crosspoints associated with each \( \sigma(M_i) \). In the case of the sparse crossbar in Figure 2.18, this count is simply \( t_1 + t_2 + t_3 + t_4 = 28 \). The sum on the right hand
side expresses the same count in terms of the numbers of $\sigma(M_i)$’s that isolate one, two and three (in general $c$) inputs. Again in the sparse crossbar of Figure 2.18, $a_1$ is isolated by $M_1, M_2$ and $M_3$, $a_2$ is isolated by $M_1, M_2$ and $M_4$, and so on. In this case, we have four inputs each of which is connected to exactly one output and isolated by three sets of three outputs, and four inputs each of which is connected to exactly two outputs and isolated by two sets of three outputs. Thus, the right hand side of the identity in Eqn. 2.26 gives $3 \times 1 \times 4 + 2 \times 2 \times 4 = 12 + 16 = 28$ which checks with our earlier count. The identity in Eqn. 2.27 can similarly be analyzed.

Moving on, we note that, when $c = m$, Eqn. 2.36 yields

$$x_s = m - (m(m-1)/n),$$

implying a lower bound of

$$nx_s = n[m - (m(m-1)/n)] = (n - m + 1)m$$

which coincides with the lower bound in Theorem 2.4. Likewise, $c = 1$ yields $x_s = 1$ leading to a lower bound of $n$ as in Eqn. 2.19.

Thus, the two lower bounds match when $c = 1$ and $c = m$. It is more cumbersome to compare the two bounds for other values of

![Figure 2.19: Comparison of Nakamura-Masson’s lower bound with the lower bound of Eqn. 2.23.](image-url)
c since Nakamura-Masson’s lower bound is implicit (it requires the solution of Eqn. 2.36.) For fixed values of $c, m,$ and $n,$ the two bounds can be compared more readily. In particular, for the $(8, 4, 2)$-concentrator shown in Figure 2.18, Nakamura-Masson’s bound gives $\lceil 8x_s \rceil = 11,$ where $x_s$ is determined as 1.3283 from Eqn. 2.36. In contrast, the lower bound of Theorem 2.6 yields 10 crosspoints. In this case, neither lower bound is tight since we established in Section 2.3.1 that a sparse crossbar $(8, 4, 2)$-concentrator must have at least 12 crosspoints. Figure 2.19 shows how the two bounds are related together for various other values of $c.$ It should be pointed out that the new lower bound gets closer to Nakamura-Masson’s lower bound as $n/m \to 1.$ To better grasp how these two lower bounds are related, the reader is asked in the exercise section to prove a set of results that show that Nakamura-Masson’s lower bound is, in general, tighter than the lower bound of Theorem 2.4. We just note that, since $x_s \leq c,$ we have $(m - x_s)!/(c - x_s)! \geq (m - c)!$, and this together with Eqn. 2.35 implies that

$$c - x_s \leq \frac{\binom{m}{c}c(c-1)}{n}, \quad (2.37)$$

or,

$$x_s \geq c - \frac{\binom{m}{c}c(c-1)}{n}. \quad (2.38)$$

If $m$ and $c$ are held fixed then $x_s \to c,$ as $n \to \infty.$ However, the right hand side of this inequality can become negative for certain fixed values of $n, m$ and $c.$ On the other hand, it is possible to obtain better bounds on $x_s$ (both upper and lower bounds) by a more careful analysis of Eqn. 2.36.

These facts aside, the lower bound given in Theorem 2.15 is much easier to compute than the Nakamura-Masson’s lower bound. The question that remains open is whether one can strengthen the former bound by using an argument along the lines given in Theorem 2.15. As a step in this direction, the reader is asked in Problem 2.36 to show that a sparse crossbar $(n, m, c)$-concentrator requires at least $2n + c - m - 2$ crosspoints. In fact, it can be shown that this linear bound is tighter than both bounds for $m \gg n - m.$
2.4.3 Optimal Sparse Crossbar Concentrators

Can we construct a sparse crossbar \((n, m, c)\)-concentrator with a minimum crosspoint complexity for any \(n, m,\) and \(c\)? Unlike the full capacity case, the resolution of this question is complicated by two related facts. First, the explicit lower bound we derived in the previous section is not as tight as Nakamura-Masson’s lower bound. Second, Nakamura-Masson’s lower bound is not explicit enough to suggest a bounded capacity sparse crossbar concentrator construction whose crosspoint complexity may somehow match it by a constant factor. Despite these obstacles, we know of some non-trivial\(^7\) sparse crossbar \((n, m, c)\)-concentrators whose crosspoint complexity either matches or is very close to the lower bounds derived in the previous section.

A sparse crossbar \((n, m, c)\)-concentrator, \(G\) will be called optimal if among all sparse crossbar \((n, m, c)\)-concentrators, none has fewer crosspoints than \(G\). In general, it is not easy to establish whether or not a sparse crossbar concentrator is optimal since the definition requires an exhaustive testing. However, if the crosspoint complexity of a sparse crossbar concentrator matches either of the lower bounds established in the previous section, or some other known lower bound, we can then easily assert its optimality. In such cases, it is also useful to define an asymptotical notion of optimality. We call a sparse crossbar concentrator asymptotically optimal if its crosspoint complexity is within a constant factor of a known lower bound.

The first family of optimal sparse crossbar concentrators was discovered by Nakamura and Masson:

**Corollary 2.18** For \(1 \leq v \leq m - 2\), the binomial \(\binom{m}{v}\)-network gives an optimal sparse crossbar concentrator with \(\binom{m}{v}\) inputs, \(m\) outputs, and a capacity of \(v + 2\).

**Proof:** That the binomial \(\binom{m}{v}\)-network is sparse crossbar concentrator for any \(v, 1 \leq v \leq m - 2\) has already been established in Theorem 2.5. Furthermore, the number of crosspoints in the binomial \(\binom{m}{v}\)-network is given by \(\binom{m}{v} v\), and Nakamura-Masson’s lower bound

\(^7\) Other than \(c = 1\) and \(c = m\) cases.
bound gives \( \binom{m}{v} \) \( x_s \) crosspoints, where \( x_s = v \) satisfies Equation 2.36 with \( n = \binom{m}{v} \), and \( c = v + 2 \). ||

This result demonstrates that Nakamura-Masson’ lower bound is tight in at least some cases, but the ratio of the number of inputs to the number of outputs of these binomial concentrators tends to 0 very rapidly as the capacity moves away from the endpoints of the domain \( \{1, 2, \ldots, m - 2\} \). The following result shows that there exist asymptotically optimal sparse crossbar concentrators for which the ratio of the number of inputs to the number of outputs tends to 1.

**Theorem 2.19** Let \( G = (I, O, E) \) be a sparse crossbar, where \( I = \{1, 2, \ldots, n\} \), \( O = \{1, 2, \ldots, m\} \), \( n - m \leq c \leq \lfloor m/c \rfloor \). Suppose that the inputs in \( I \) are partitioned into two sets

\[
I_1 = \{1, 2, \ldots, n - c\} \quad \text{and} \quad I_2 = \{n - c + 1, n - c + 2, \ldots, n\},
\]

and the first \( \lfloor m/c \rfloor \) outputs in \( O \) are partitioned into \( \lfloor m/c \rfloor \) sets,

\[
O_i = \{(i - 1)c + 1, (i - 1)c + 2, \ldots, ic\}, 1 \leq i \leq \lfloor m/c \rfloor.
\]

Let the inputs in \( I_2 \) be connected to the outputs in each of \( O_1, O_2, \ldots, O_{\lfloor m/c \rfloor} \) in a diagonal fashion, i.e., let \( (n - c + j, (i - 1)c + j) \in E, 1 \leq i \leq \lfloor m/c \rfloor, 1 \leq j \leq c \). Furthermore, let input \( i \) in \( I_1 \) be connected to output \( i \) in \( O, 1 \leq i \leq n - c \). Then \( G \) (See Figure 2.20) is an \((n, m, c)\)-concentrator with \( n - c + c \lfloor m/c \rfloor \) crosspoints.

**Proof:** We need to show that every subset of \( c \) inputs can be matched with some \( c \) outputs, \( n - m \leq c \leq \lfloor m/c \rfloor \). It is obvious from the construction that if the \( c \) inputs all belong to \( I_1 \) or they all belong to \( I_2 \) then this can be done. So, consider an arbitrary but fixed set of \( \alpha \leq c \) inputs, \( X \), where \( X_1 = X \cap I_1 \neq \emptyset \), and \( X_2 = X \cap I_2 \neq \emptyset \). Let \( \alpha_1 = |X_1|, \alpha_2 = |X_2| \), and let \( Y_1 \) and \( Y_2 \) be the sets of neighbors of \( X_1 \) and \( X_2 \) in \( G \). We have

\[
|Y_1 \cup Y_2| = |Y_1| + |Y_2| - |Y_1 \cap Y_2|,
\]

and by the construction of \( G \), \( |Y_1| = \alpha_1, |Y_2| \geq \alpha_2 \lfloor m/c \rfloor, |Y_1 \cap Y_2| \leq \min\{\alpha_1, \alpha_2 \lfloor m/c \rfloor\} \) so that

\[
|Y_1 \cup Y_2| \geq \alpha_1 + \alpha_2 \lfloor m/c \rfloor - \min\{\alpha_1, \alpha_2 \lfloor m/c \rfloor\}.
\]
Now, since $\alpha_2 \geq 1$, and $c \leq \lfloor m/c \rfloor$, we have

$$\alpha_2 \lfloor m/c \rfloor \geq \lfloor m/c \rfloor \geq c \geq \alpha_1.$$ 

Hence,

$$|Y_1 \cup Y_2| \geq \alpha_1 + \alpha_2 m/c - \alpha_1 = \alpha_2 \lfloor m/c \rfloor \geq c \geq \alpha.$$

We have thus shown that the set of neighbors of any $\alpha$ inputs contains at least $\alpha$ outputs. Hence by Hall’s theorem, $\mathcal{G}$ is an $(n, m, c)$-concentrator for any $c, n - m \leq c \leq \sqrt{m}$. Moreover, its construction reveals that it encompasses $n - c + \lfloor m/c \rfloor c$ crosspoints, where the first term accounts for the number of crosspoints connected to the inputs in $I_1$, and the second term accounts for the number of crosspoints connected to the inputs in $I_2$. ||

Let us further scrutinize this construction. The rationale behind restricting the values of $c$ between $n - m$ and $\sqrt{m}$ has much to do
with the structure of the construction. In particular, the inequality \( n - m \leq c \) coupled with the diagonal pairing of the inputs in \( I_1 \) with the \( m \) outputs permits up to \( n - c \) of the inputs in \( I_1 \) to have a matching set of outputs. If this inequality is relaxed then two or more inputs in \( I_1 \) will be forced to share the same output(s) as neighbors, resulting in a more complex construction. On the other hand, the inequality \( c \leq \sqrt{m} \) is put in place to prevent an input in \( I_2 \) from being blocked in case one chooses to concentrate \( c - 1 \) inputs in \( I_1 \) in such a way that every one of these inputs shares an output with that input. For example, consider the sparse crossbar construction in Figure 2.21, where \( n = 12, m = 10, \) and \( c = 3. \) The inequality \( c \leq \sqrt{m}, \text{or} \ c \leq m/c \) ensures that each input in \( I_2 \) is connected to one more output than the total number of outputs that can be blocked by as many as \( c - 1 \) inputs in \( I_1. \) For example, any two of the first, fourth and eighth inputs in \( I_1 \) can block the first input in \( I_2 \) from reaching at most two of its three neighbors, but no more. Hence, the first (as well any other) input in \( I_2 \) can still be matched with an output even when it is combined with any two inputs in \( I_1 \) in an assignment.

The crosspoint complexity of this sparse crossbar concentrator construction is very close to the lower bound derived in the previous section. More specifically, it remains within a factor of two of that lower bound. To see this, we just note that

\[
n - c + \left\lfloor \frac{m}{c} \right\rfloor c \leq n - c + m \leq \left( \frac{n - c}{m - c + 1} + 1 \right) m.
\]

For a more specific comparison, we note that the construction in Figure 2.21 uses \( 9 + 9 = 18 \) crosspoints, whereas the lower bound yields \( \left\lfloor \frac{(n - c + 1)m/(m - c + 1)}{10 \times 10/8} \right\rfloor = 11 \) crosspoints\(^8\). We also note that a more direct construction of a sparse crossbar \((n, m, c)\)-concentrator obtained by connecting each input to some \( c \) outputs yields \( nc = O(n^{1.5}) \) crosspoints as compared to \( O(n) \) crosspoints of this construction for \( n - m \leq c \leq \sqrt{m} \).

In fact, we can do better, for arbitrary \( n, m, \) and \( c, \) if we extend the construction in Theorem 2.7.

\(^8\) The lower bound we mentioned at the end of Section 2.4.2 gives 15 crosspoints.
Theorem 2.20 Let $G$ be a sparse crossbar with $n$ inputs and $m$ outputs, where each of any fixed $n - m$ inputs is connected to exactly $c$ outputs and each of the remaining $m$ inputs is connected to a distinct output. Then $G$ is an $(n, m, c)$-concentrator with $cn - m(c - 1)$ crosspoints.

Proof: It is left as an exercise.

In connection with the construction of optimal sparse crossbar concentrators, let us also mention an extension of the fat-and-slim crossbar.

Definition 2.6 Let $n \geq m, p$ be positive integers such that $p$ divides both $m$ and $n - m$. Let $G = (I, O, E)$ be a sparse crossbar. Let $I = \{1, 2, \ldots, n\}$ be partitioned into $p + 1$ sets of inputs, $I_1, I_2, \ldots, I_{p+1}$, where $|I_i| = (n-m)/p, 1 \leq i \leq p, |I_{p+1}| = m$, and $O = \{1, 2, \ldots, m\}$. $G$ is called a fat-and-slim crossbar with dilation $p$ if
(i) each of the inputs in $I_i$ is connected to outputs $i, i+p, i+2p, \ldots, i+(\frac{m-p}{p})p$,
(ii) each of the inputs in $I_{p+1}$ is connected to a distinct output.

The following result holds for dilated fat-and-slim crossbars.

**Theorem 2.21** An $(n, m)$ fat-and-slim crossbar with dilation $p$ is an $(n, m, m/p)$-concentrator with $m((n - m)/p + 1)$ crosspoints.

**Proof:** Let $X$ be an arbitrary but fixed subset of $m/p$ inputs and define $X_i = X \cap I_i, 1 \leq i \leq p+1$. Obviously, $|X_1| + |X_2| + \ldots |X_{p+1}| = m/p$. Let $N(X_i) = \{b \in O : (x, b) \in E, \text{ for some } x \in X_i\}, 1 \leq i \leq p+1$. By the construction of $\mathcal{G}$, it is easy to see that $N(X_i) \cap N(X_j) = \emptyset$ for any $i, j, 1 \leq i \neq j \leq p$. Thus, the cardinality of the union of outputs that are incident upon the inputs in $X$ must be at least $m/p$. Therefore, by Hall’s Theorem, $X$ has a matching set. ||

A fat-and-slim crossbar with 18 inputs, 6 outputs, and dilation 3 is shown in Figure 2.22.

Interestingly enough, there is a close relation between the crosspoint complexity of a fat-and-slim crossbar with dilation $p$ and the lower bound of Theorem 2.15. Recall that, upon simplifying the combinatorial terms, this lower bound reduces to

$$\left\lfloor \frac{(n-c+1)m}{m-c+1} \right\rfloor = \left\lfloor m\left(\frac{n-m}{m-c+1} + 1\right)\right\rfloor.$$ 

Now, if we replace $c$ by $m - p + 1$ then the lower bound becomes

$$\left\lfloor m\left(\frac{n-m}{p} + 1\right)\right\rfloor.$$ 

In other words, the crosspoint complexity of a fat-and-slim crossbar with dilation $p$ matches the lower bound of Theorem 2.15 on the crosspoint complexity of an $(n, m, m - p + 1)$-concentrator!! Since $m - p + 1 \geq m/p$ for all $p, m \geq 2$ we conclude that the fat-and-slim crossbar with dilation $p$ is not optimal with respect to the lower bound of Theorem 2.15. In a way, the gap between the capacities $m - p + 1$ and $m/p$ illustrates the difficulty of constructing an optimal or asymptotically optimal $(n, m, c)$-concentrator for an arbitrary $c, 1 < c < m$. In the next chapter we will introduce a seminal result of Pinsker that puts to rest the question of the existence of $(n, m, c)$-concentrators with $O(n)$ crosspoints.
2.4.4 Large Capacity Sparse Crossbars

We have shown in the previous section that a \((2m, m, \sqrt{m})\) sparse crossbar concentrator can be constructed with as few as \(2m\) crosspoints. The question that naturally arises is whether or not the capacity can be increased any further without increasing the crosspoint count beyond some small multiple of \(m\). In particular, does there exist a sparse crossbar \((m + \epsilon, \alpha m, m)\)-concentrator, with \(\beta m\) crosspoints, where \(\epsilon > 0\), \(0 < \alpha < 1\) and \(\beta\) is a small positive constant? We will show that the answer is in the affirmative by constructing a \((m + \sqrt{m}, m/2)\)-sparse crossbar with \(3m\) crosspoints.

Let \(G = (I, O)\) be a \((m + d\sqrt{m}, m)\)-sparse crossbar, where \(d|\sqrt{m}\).

Let \(I\) be partitioned into

\[
I_j = \{(j,1), (j,2), \ldots, (j,d\sqrt{m})\}, 1 \leq j \leq \frac{\sqrt{m}}{d} + 1;
\]

\(O\) be partitioned into

\[
O_j = \{(j,1), (j,2), \ldots, (j,d\sqrt{m})\}, 1 \leq j \leq \frac{\sqrt{m}}{d}.
\]

Suppose that input \((j, k), 1 \leq j \leq \frac{\sqrt{m}}{d}, 1 \leq k \leq d\sqrt{m}\), is connected to outputs \((j, k)\), and \((j, (k+1) \mod (d\sqrt{m}))\); input \((\frac{\sqrt{m}}{d} + 1, k), 1 \leq k \leq \frac{\sqrt{m}}{d}\), and \((\frac{\sqrt{m}}{d}, (k+1) \mod (d\sqrt{m}))\) are connected to outputs \((\frac{\sqrt{m}}{d}, k)\), and \((\frac{\sqrt{m}}{d} + 1, (k+1) \mod (d\sqrt{m}))\) respectively.
$d \sqrt{m}$, is connected to outputs $(r, k)$, $r = 1, 2, \ldots, \frac{\sqrt{m}}{d}$. The structure of this sparse crossbar is illustrated in Figure 2.23, where parameter $d$ is introduced to capture the relation between the capacity and the input/output ratio of a sparse crossbar concentrator. We will show that $c \geq \frac{m}{d^2} - \frac{2d}{\sqrt{m}} + 1$.

We begin with the following lemmas wherein it is assumed that $R$ is a subset of $\alpha \leq m$ inputs, and $R_j = R \cap I_j, 1 \leq j \leq \frac{\sqrt{m}}{d} + 1$.

**Lemma 2.22** If $R_j \neq I_j$, and $R_j \neq \emptyset$, $1 \leq j \leq \frac{\sqrt{m}}{d} + 1$, then $|N(R_j)| \geq |R_j| + 1$.

**Proof:** Let $R_j = \{(j, r_1), (j, r_2), \ldots, (j, r_p)\}$. Since $R_j \neq I_j$, and $R_j \neq \emptyset$, we must have $1 \leq p < d \sqrt{m}$. Thus there exist $(j, r_k) \in R_j$ such that $(j, (r_k + 1) \mod d \sqrt{m}) \notin R_j$. Since each input in $R_j$ is connected to a distinct output in $N(R_j)$, and $(j, r_k)$ is also connected to $(j, (r_k + 1) \mod d \sqrt{m})$, the statement follows. ||

**Lemma 2.23** If any $R_j = \emptyset$, $1 \leq j \leq \frac{\sqrt{m}}{d} + 1$, then $|N(R)| \geq \alpha$. 

![Figure 2.23: A bounded capacity concentrator with parameter $d$.](image)
Proof:

1. If \( j = \frac{\sqrt{m}}{d} + 1 \), then input \((i, k), 1 \leq k \leq d\sqrt{m}, 1 \leq i \leq \frac{\sqrt{m}}{d}\) is connected to a distinct output \((i, k)\). Thus we have \( |N(R)| \geq \alpha \).

2. If \( j \neq \frac{\sqrt{m}}{d} + 1 \), then input \((i, k), 1 \leq k \leq d\sqrt{m}, 1 \leq i \neq j \leq \frac{\sqrt{m}}{d}\), is connected to a distinct output \((i, k)\), and input \((\frac{\sqrt{m}}{d} + 1, k)\) is connected to a distinct output \((j, k)\), \(1 \leq k \leq d\sqrt{m}\). Thus we again have \( |N(R)| \geq \alpha \). ||

Lemma 2.24 If \( |R_{\frac{\sqrt{m}}{d} + 1}| \geq \frac{d\alpha}{\sqrt{m}} \), then \( |N(R)| \geq \alpha \).

Proof: Since every input in \( R_{\frac{\sqrt{m}}{d} + 1} \) is connected to \( \frac{\sqrt{m}}{d} \) distinct outputs, we have \( |N(R)| \geq |N(R_{\frac{\sqrt{m}}{d} + 1})| \geq |R_{\frac{\sqrt{m}}{d} + 1}| \times \frac{\sqrt{m}}{d} \geq \alpha \). ||

Theorem 2.25 \( G_2 \) is a \((m + d\sqrt{m}, m, c)\)-sparse crossbar concentrator, where \( c \geq m/d^2 - \frac{2d}{\sqrt{m}} + 1 \).

Proof:

1. If \( R_j = \emptyset \) for any \( j, 1 \leq j \leq \frac{\sqrt{m}}{d} + 1 \), then it follows from Lemma 2.23 that we have \( |N(R)| \geq \alpha \).

2. If \( R_{\frac{\sqrt{m}}{d} + 1} \geq \frac{d\alpha}{\sqrt{m}} \), then from Lemma 2.24 we have \( |N(R)| \geq \alpha \).

3. In the case that \( R_j \neq \emptyset, 1 \leq j \leq \frac{\sqrt{m}}{d} + 1, \) and \( |R_{\frac{\sqrt{m}}{d} + 1}| < \frac{dk}{\sqrt{m}} \), let \( p = |R_{\frac{\sqrt{m}}{d} + 1}| \), and \( q \) be the number of sets \( I_j \) such that \( I_j \neq R_j \), \( 1 \leq j \leq \frac{\sqrt{m}}{d} \). Since \( |I_j| = d\sqrt{m} \), and each of \( R_j, 1 \leq j \leq \frac{\sqrt{m}}{d} \), has at least one element, we have

\[
q \geq \frac{\sqrt{m}}{d} - \frac{\alpha - p - \frac{\sqrt{m}}{d}}{d\sqrt{m} - 1} \tag{2.39}
\]

From Lemma 2.22, we know that

\[|N(R)| \geq \alpha - p + q,\]
and so if \( q \geq p \) then \( |N(R)| \geq \alpha \). Now, by Eqn. (2.39), \( q \geq p \), if

\[
\frac{\sqrt{m}}{d} - \frac{\alpha - p - \frac{\sqrt{m}}{d}}{d\sqrt{m} - 1} \geq p,
\]

or

\[
\frac{\sqrt{m}}{d} - \frac{\alpha - \frac{\sqrt{m}}{d}}{d\sqrt{m} - 1} \geq p(d\sqrt{m} - 2).\]

Since \( p \leq \frac{d\alpha}{\sqrt{m}} \), the latter inequality holds if

\[
\frac{\sqrt{m}}{d} - \frac{\alpha - \frac{\sqrt{m}}{d}}{d\sqrt{m} - 1} \geq \frac{d\alpha}{\sqrt{m}} \times \frac{d\sqrt{m} - 2}{d\sqrt{m} - 1},
\]

or

\[
\frac{\sqrt{m}}{d} + \frac{\sqrt{m}}{d\sqrt{m} - 1} \geq \frac{\alpha}{d\sqrt{m} - 1} (d^2 + 1 - \frac{2d}{\sqrt{m}}),
\]

or

\[
m - \frac{\sqrt{m}}{d} + \frac{\sqrt{m}}{d} \geq \alpha (d^2 + 1 - \frac{2d}{\sqrt{m}}),
\]

or

\[
\alpha \leq \frac{m}{d^2 + 1 - \frac{2d}{\sqrt{m}}}. \tag{2.40}
\]

Thus, if \( \alpha \leq m/d^2 - \frac{2d}{\sqrt{m}} + 1 \) then \( |N(R)| \geq \alpha \).

Combining the three cases with Hall Theorem proves the statement.

The following is a direct corollary of this theorem.

**Corollary 2.26** A sparse crossbar \( (m + \sqrt{m}, m, m/2) \) concentrator can be constructed using \( 3m \) crosspoints.

In fact the construction in Theorem 2.25 provides a family of sparse crossbar concentrators as \( d \) is varied between 1 and \( \sqrt{m} \). The concentrator in the corollary has the largest capacity, but also the smallest
ratio of the number of inputs to number of outputs in the family. At the other extreme is a sparse crossbar \((2m, m, \sqrt{m})\)-concentrator.

The capacity of the sparse crossbar construction in Theorem 2.25 can be increased by adding another \(m\) crosspoints as shown in Figure 2.24. We leave it to the reader to prove that the capacity of this sparse crossbar construction is at least

\[
\min \left( \frac{m}{d^2}, \frac{m}{2d} \times \left( \frac{1}{\sqrt{m}} - \frac{1}{d} + \sqrt{2 + \frac{1}{m} - \frac{1}{d}} \right) \right).
\]

Let us conclude this section by mentioning how the following much acclaimed König’s theorem (sparse crossbar version) can be used to upper bound the capacity of a sparse crossbar.

**Theorem 2.27 (König)** Let \(\mathcal{G} = (I, O, E)\) be a sparse crossbar. The maximum number of crosspoints in a matching is given by the minimum number of rows and columns that contain all of the crosspoints of \(\mathcal{G}\).

**Proof:** It is left as an exercise.
To see how König’s theorem can be used to upper bound the capacity of a sparse crossbar, consider the sparse crossbar in Figure 2.25. Since its crosspoints are contained in no more than two rows and a column, by König’s Theorem, it has a capacity of at most 3. Strikingly, this sparse crossbar uses more crosspoints than the achievable lower bound (Theorem 2.4) for a full capacity (four) sparse crossbar concentrator.

It should be noted that an \((n,m)\)-sparse crossbar does not necessarily have capacity \(m\), if no fewer than \(m\) of its rows and columns contain all of its crosspoints. If this is the case, König’s theorem only implies that some \(m\) inputs of that crossbar have a matching set. For a sparse crossbar to have capacity \(m\), every \(m\) of its inputs must have a matching set, not just one. Of course, we can apply König’s Theorem iteratively to determine if a sparse crossbar has a capacity \(m\). This entails applying the theorem to each subgraph of \(G\) whose outputs coincide with those of \(G\) and whose inputs coincide with an \(m\)-subset of inputs of \(G\). This essentially amounts to applying Hall’s theorem for the same purpose.

Implicit in König’s theorem is a necessary condition for an \((n,m)\)-sparse crossbar to have capacity \(m\). A number of corollaries of König (or Hall’s) theorem can be used to obtain sufficient conditions. Here we state one such result. Its proof in graph theory language can be
found in C. Berge's classic text on graphs [Ber85].

**Theorem 2.28** Let \( G = (I, O, E) \) be an \((n, m)\)-sparse crossbar. For any \( m \)-subset \( X \subset I \), index the inputs \( x_j \in X \) such that

\[ \text{Deg}(x_1) \geq \text{Deg}(x_2) \geq \ldots \geq \text{Deg}(x_m) \]

and likewise, index the outputs \( y_j \in O \) such that

\[ \text{Deg}(y_1) \leq \text{Deg}(y_2) \leq \ldots \leq \text{Deg}(y_m), \]

where the degrees are computed with respect to the graph \( G_X = (X, O, E_X) \). Then \( G \) is an \((n, m)\)-concentrator, if, for every \( m \)-subset \( X \) of \( I \), we have \( \text{Deg}(y_1) > 0 \), and

\[
\sum_{j=1}^{k} \text{Deg}(y_j) > \sum_{j=1}^{k-1} \text{Deg}(x_j); \ k = 2, 3, \ldots, m.
\]

To see how this result can be used to determine whether a given \((n, m)\)-sparse crossbar has capacity \( m \), consider the sparse crossbar construction shown in Figure 2.26. By inspection, it is seen that the degree of every input is 3. Furthermore, it can be verified that the degrees of the outputs in any fixed 4-subset of inputs, when ordered from the smallest to the largest match one of the following three patterns: 2244, 2334, and 3333. Therefore, the inequality between the sums in the corollary is satisfied for \( k = 2, 3, 4 \) so that this graph is a \((6, 4)\)-concentrator.

### 2.4.5 Deficiency of a Sparse Crossbar

In certain concentrator designs it is important to know the maximum number of inputs that can be matched to outputs when the inputs and outputs must meet certain design constraints. For example, consider the following problem. A computer switch board is to be assembled to grant incoming login requests over telephone lines to a main frame computer via a set of modems. There is a total of 100 telephone lines (and hence 100 users) that are divided into four
groups of 25 lines, $A$, $B$, $C$, and $D$. The switch board houses seventy-five modems so that each line in group $A$ can be connected to any one of 20 of these modems, each line in group $B$ can be connected to any one of 10 of these modems, each line in group $C$ can be connected to any one of 4 of these modems, and each line in group $D$ can be connected to one of 2 of these modems. Each modem can be accessed (in tandem) by at most 10 telephone lines. If all the users on all 100 telephone lines attempt to login at once, what is the minimum number of users that will be able to login regardless of the actual crosspoints between the telephone lines and the modems?

We can solve this problem by modeling the connections between the telephone lines and the modems as a sparse crossbar with 100 inputs (users) and 75 outputs (modems). Potentially, up to 75 users can login all at once, but this is not very likely to happen because of the stated in-degree and out-degree constraints. To determine the minimum number of users that can login all at once, we can utilize a notion, called the deficiency of a sparse crossbar.

Let $G = (I, O, E)$ be a sparse crossbar, and let $X$ be a subset of inputs of $G$. The deficiency $\vartheta(X)$ of $X$ is defined as $\vartheta(X) = |X| - |N(X)|$, where $N(X) = \{ b \in O : (x, b) \in E \text{ for some } x \in X \}$. The deficiency
of $\vartheta(G)$ of $G$ is defined by $\vartheta(G) = \text{Max}_{X \subseteq I} \{ \vartheta(X) \}$. It is easy to see that the deficiency of a subset of inputs can be negative, zero, or positive while the deficiency of a sparse crossbar is always a non-negative number (the deficiency of the empty set = 0).

We state the following well-known theorem from graph theory without a proof (See, for example [Liu68]).

**Theorem 2.29** In a sparse crossbar $G = (I, O, E)$, the maximum number of inputs in $I$ that can be matched with the outputs in $O$ is given by $|I| - \vartheta(G)$.

Thus, if we determine the maximum deficiency of the sparse crossbar by which we model the computer switch board then invoking this theorem will give us a lower bound on the number of users that can all login at once. Now, the deficiency of a subset of inputs $X$ which contains $u_1$ users from the first group, $u_2$ users from the second group, $u_3$ users from the third group and $u_4$ users from the fourth group is given by

$$\vartheta(X) = |X| - |N(X)| = u_1 + u_2 + u_3 + u_4 - |N(X)|.$$

But, since each user in the first group can connect to any one of 20 modems, each user in the second group can connect to any one of 10 modems, each user in the third group can connect to any one of 4 modems, and each user in the fourth group can connect to either one of 2 modems, and no more than 10 users are connected to any given modem, we have

$$|N(X)| \geq \frac{20u_1 + 10u_2 + 4u_3 + 2u_4}{10}.$$

It follows that

$$\vartheta(X) \leq u_1 + u_2 + u_3 + u_4 - \frac{20u_1 + 10u_2 + 4u_3 + 2u_4}{10}$$

or

$$\vartheta(X) \leq \frac{-10u_1 + 6u_3 + 8u_4}{10}.$$  

Noting that the maximum value of $\vartheta(X)$ occurs when $u_1 = 0$, and $u_2 = 25$, and $u_4 = 25$, we have $\vartheta(G) \leq 35$, so that, by the above
BOUNDED CAPACITY SPARSE CROSSBARS

theorem, the minimum number of users that can login at once under
the stated fanin and fanout restrictions is 100 − 35 = 65.

This sounds too good to be true given that there are only 75 modems,
and each modem is accessible by at most ten lines. The point is that
not every group of 65 users can login at once. It is just that there is
at least one such group. So, the deficiency of a sparse crossbar con-
centrator is a “weaker” notion than its capacity. A sparse crossbar
with capacity $c$ permits any $c$ inputs to have a matching with some $c$
outputs whereas a sparse crossbar $G = (I, O, E)$ with deficiency $\vartheta(G)$
guarantees only one subset of $|I| - \vartheta(G)$ inputs to have a matching
with an equal number of outputs. There may possibly be more such
subsets of inputs, but the upper bound on the deficiency of $G$ of-
fers no guarantees on that. On the contrary, it is quite likely that
there exist subsets of inputs with cardinality $\leq |I| - \vartheta(G)$ without
any matching sets. While there exist some 65 users that can login
at once, it is also quite possible that a much smaller group of users
cannot login all at once.

2.5 TOPOLOGICAL ISSUES

So far our study of concentrators has been dominated by the cross-
point complexity and capacity issues. In this main section, we will
examine certain set, graph and matrix theoretical characterizations
of sparse crossbars, and topological equivalences among them. Topo-
logical equivalences amongs sparse crossbar concentrators are worth
studying in their own right, but they are also useful in determining
the distribution and density of such graphs among all sparse cross-
bars. We will deal with this question after we formally introduce the
notion of topological equivalence in the next section.

2.5.1 Equivalent Sparse Crossbar Concentrators

For any two sparse crossbar concentrators to be topologically equiva-
 lent, it is reasonable to require that for each input (output) with any
given out-degree (in-degree) in one sparse crossbar, there be an in-
put (output) with the same out-degree (in-degree) in the other sparse
crossbar. Additionally, we would expect that the edges (crosspoints)
between inputs and outputs are preserved between two topologically equivalent sparse crossbars. We formalize these requirements as follows.

**Definition 2.7** Sparse crossbar concentrators $G_1 = (I_1, O_1, E_1)$ and $G_2 = (I_2, O_2, E_2)$ are said to be topologically equivalent (isomorphic) if there exist two bijections $\rho : I_1 \rightarrow I_2$ and $\gamma : O_1 \rightarrow O_2$ such that $(\rho(a), \gamma(b)) \in E_2$ if and only if $(a, b) \in E_1$. If it exists, the pair of bijections $(\rho, \gamma)$ is called an isomorphism from $G_1$ onto $G_2$, and if $G_1 = G_2$ then $(\rho, \gamma)$ is called an automorphism of $G_1$.

Loosely speaking, the two maps $\rho$ and $\gamma$ merely reorder the rows and the columns of $G_1$ so that $G_1$ looks exactly as $G_2$. This is illustrated in Figure 2.27 for two binomial concentrators. The concentrator on the bottom righthand corner is obtained by swapping the 2nd and 3rd rows, and 2nd and 3rd columns of the concentrator on the left ($\rho(a_2) = a_3, \rho(a_3) = a_2, \gamma(b_2) = b_3, \gamma(b_3) = b_2$). It should be noted that the order in which the rows and columns are swapped makes no difference in mapping $G_1$ onto $G_2$ as demonstrated in the figure.

In general this is true for any isomorphism $(\rho, \gamma)$ between two sparse crossbar concentrators. To see this, consider a crosspoint at the intersection of any given row $x$ and column $y$. If row $x$ and column $y$ are moved to row $x'$ and column $y'$ by $\rho$ and $\gamma$, respectively that crosspoint ends up at the intersection of row $x'$ and column $y'$ regardless of whether we moved row $x$ to row $x'$ first and then column $y$ to column $y'$, or the other way around. Another way to think about this is that the row and column transformations on a sparse crossbar $G$ can be viewed as a product of three matrices $R, A_G, C$ in that order, where $R$ and $C$ are $m \times m$ and $n \times n$ permutation matrices, respectively, corresponding to $\rho$ and $\gamma$, and $A_G$ is the adjacency matrix of $G$. The associativity of matrix multiplication then dictates that $RA_GC$ can be carried out either as $(RA_G)C$ or as $R(A_GC)$. For example, in Figure 2.27, we have

$$A_{G_1} = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}, A_{G_2} = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \end{bmatrix},$$
and

\[
R = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix},
\quad C = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}
\]

so that \((RA_{\mathcal{G}_1})C = R(A_{\mathcal{G}_1}C) = A_{\mathcal{G}_2}\).

In essence, an isomorphism between two sparse crossbar concentrators demonstrates an ability on the part of each concentrator to “simulate” the concentration assignments of the other, and conversely. For example, the subset of inputs, \(\{a_1, a_2\}\) of \(\mathcal{G}_1\) has two matching sets, namely, \(\{b_1, b_2\}\) and \(\{b_1, b_3\}\). Relying on the isomorphism between \(\mathcal{G}_1\) and \(\mathcal{G}_2\), we see that \(\{a_1, a_2\}\) can be matched with either \(\{b_1, b_2\}\) or \(\{b_1, b_3\}\), by relabeling \(a_2\) as \(a_3\); \(a_3\) as \(a_2\); \(b_2\) as \(b_3\); and \(b_3\) as \(b_2\) in \(\mathcal{G}_2\).

A key question on topologically equivalence is the enumeration of all isomorphic copies of a given sparse crossbar concentrator. This problem can be reduced to determining the number of automorphisms of a sparse crossbar concentrator. To see this, let \((\rho_1, \gamma_1)\) and \((\rho_2, \gamma_2)\) be any two automorphisms of a sparse crossbar concentrator \(\mathcal{G} = (I, O, E)\). It is immediate that their composition \((\rho_1 \rho_2, \gamma_1 \gamma_2)\), defined by \((\rho_1(\rho_2(a)), \gamma_1(\gamma_2(b)))\), where \(a \in I\) and \(b \in O\) is also an automorphism of \(\mathcal{G}\). Furthermore, for any automorphism \((\rho, \gamma)\) of \(\mathcal{G}\), the inverse of \((\rho, \gamma)\) defined by \((\rho^{-1}(a), \gamma^{-1}(b))\), where \(a \in I\) and \(b \in O\), is also an automorphism of \(\mathcal{G}\). Given these facts, and that the identity map \((e_I, e_O)\) is also an automorphism of \(\mathcal{G}\), where \(e_I(a) = a\) for all \(a \in I\) and \(e_O(b) = b\) for all \(b \in O\), the set of all automorphisms, \(\text{Aut}(\mathcal{G})\), of \(\mathcal{G}\) forms a group, called the automorphism group of \(\mathcal{G}\).

Now let \(\Sigma_I\) and \(\Sigma_O\) be the symmetric groups of permutations acting on the sets of inputs and outputs, \(I\) and \(O\), respectively. Consider the set, \(S_{I,O}\) of all pairs \((\gamma, \rho)\) where \(\gamma \in \Sigma_I\) and \(\rho \in \Sigma_O\). The set \(S_{I,O}\) is the cartesian product of two symmetric groups which obviously forms a group of \(|I||O|\) elements under the component-wise composition of its maps. Furthermore, it can be shown that \(\text{Aut}(\mathcal{G})\) forms a subgroup of \(S_{I,O}\). The right (or left) cosets of \(\text{Aut}(\mathcal{G})\)
then partition the set of all sparse crossbar concentrators that are obtained by permuting the inputs and outputs of $G$. More specifically, it can be shown that all permutations in each coset map $G$ to the same isomorphic copy. Consequently, we have the following fact.

**Theorem 2.30** Let $G = (I, O, E)$ be a sparse crossbar concentrator. The number of isomorphic copies of $G$ is given by $|I|!|O|!/|\text{Aut}(G)|$.

Let us illustrate this result on the $\binom{3}{2}$-binomial network. In this case $|I| = |O| = 3$ and hence $S_{I,O}$ contains $3!3! = 36$ permutations. By the inspection of the network, it is immediate that no permutation
(γ, e_O) or (e_I, ρ), where γ ∈ Σ_I − {e_I} and ρ ∈ Σ_O − {e_O}, can be in the automorphism group of this network. This leaves 36 − 10 = 26 permutations as likely automorphisms of the network. Clearly, the identity (e_I, e_O) is one of these automorphisms. Each of the remaining automorphisms must fix the \( \binom{3}{2} \) network to itself as shown in Figure 2.28(a). It can be shown that there are exactly six automorphisms, including the identity which form the group shown in Figure 2.28(b)\(^9\). By the previous theorem, these generate the six isomorphic copies of the \( \binom{3}{2} \)-network. These along with the \( \binom{3}{2} \)-network are shown in Figure 2.29.

Each of the isomorphic copies is obtained by permuting the inputs and outputs of the \( \binom{3}{2} \)-network by the maps in a distinct coset of its automorphism group. For example, the second network in the first row in Figure 2.29 can be obtained by permuting the inputs and outputs of the first network in the same row using any of the maps in the coset \((a_2 a_3), e_B)\text{Aut}(G)\). The reader should show that the other copies are similarly obtained.

In general, it is difficult to derive an exact formula for the cardinality of the automorphism group of a sparse crossbar. The relation between the cardinality of the automorphism group and the number of isomorphic copies of a sparse crossbar is useful only if the automorphism group can be determined.

When the number of isomorphic copies of a given sparse crossbar cannot be counted exactly, it is useful to bound it. The following result achieves such a bound for a binomial \( \binom{m}{2} \)-network.

**Theorem 2.31** The number of isomorphic copies of an \( \binom{m}{2} \)-network is not less than \((m − 1)!/(m − 2)! \ldots 3!\).

**Proof:** Let the number of isomorphic copies of a \( \binom{m−1}{2} \)-network be denoted by \( C_{m−1} \). From each copy of such a network we can generate at least \((m − 1)! \) isomorphic copies of the \( \binom{m}{2} \) network simply by adding \( m − 1 \) new inputs and one new output to it and pairing the inputs with outputs.

\(^9\) For simplicity of notation, the automorphisms of the \( \binom{3}{2} \)-binomial network are expressed as permutations of the indices of its inputs and outputs.
(a) The graphical depiction of the automorphism $(\gamma_2, \rho_1)$.

(b) Automorphism group of $\binom{3}{2}$-network.

Figure 2.28: The automorphisms of the binomial $\binom{3}{2}$ network.
new inputs onto the \( m \) outputs in \((m - 1)!\) ways. We conclude that \( C_m \geq (m - 1)!C_{m-1} \), and by iterating this formula until \( m = 4 \), and noting that \( C_3 = 3! \), we obtain the stated bound.

A frequently encountered problem in equivalence studies is to determine if any two graphs are isomorphic under certain structure preserving maps. In general, determining whether or not any two graphs are topologically equivalent (isomorphic) is one of a few graph theoretical problems whose complexity is not well-understood. In certain cases, however, in particular, for sparse crossbars (bipartite graphs) with certain topological features, this question can be settled within a polynomial order of time. One such feature is that the sparse crossbars under examination for topological equivalence can be expressed as a direct sum of vertex disjoint open and closed chains (cycles) as shown in Figure 2.30. It is immediate from the highlighted decompositions that the two sparse crossbars are topologically equivalent and the equivalence is obtained by mapping the cycle \((a_1, b_1, a_3, b_4, a_5, b_2)\) in \( G_1 \) to the cycle \((a_2, b_1, a_3, b_3, a_5, b_4)\) in \( G_2 \), and the open chain \((a_2, b_3, a_6, b_5, a_4)\) in \( G_1 \) to the open chain \((a_1, b_2, a_4, b_5, a_6)\) in \( G_2 \). These maps induce the row and column permutations \( \rho, \gamma \) which are given by

\[
\begin{align*}
\rho(a_1) &= a_2, \, \rho(a_2) = a_1, \, \rho(a_3) = a_3, \\
\rho(a_4) &= a_6, \, \rho(a_5) = a_5, \, \rho(a_6) = a_4,
\end{align*}
\]

and

\[
\begin{align*}
\gamma(b_1) &= b_1, \, \gamma(b_2) = b_4, \, \gamma(b_3) = b_2, \, \gamma(b_4) = b_3, \, \gamma(b_5) = b_5.
\end{align*}
\]

It is obvious that the ordered pair \((\rho, \gamma)\) defines an isomorphism from \( G_1 \) to \( G_2 \).

This example can be generalized to the following statement.

**Theorem 2.32** Two sparse crossbars \( G_1 \) and \( G_2 \) with vertex disjoint decompositions of open and closed chains are isomorphic if and only if there exists a map from the decomposition of \( G_1 \) onto the decomposition of \( G_2 \) that preserves both the types (open or closed) and lengths of their chains.
Figure 2.29: The $\binom{3}{2}$-binomial network, and its isomorphic copies.
Figure 2.30: Detecting the topological equivalence between $G_1$ and $G_2$ by examining their vertex disjoint open and closed paths.

In the above example, $G_1$ and $G_2$ each have one open chain of length 5 and one closed chain of length 6, and hence they must be topologically equivalent.

It is easy to see that any sparse crossbar with $n$ inputs and $m$ outputs can be decomposed into a direct sum of vertex disjoint chains in $O(n)$ time provided that such a decomposition does exist. Once such a decomposition for each of $G_1$ and $G_2$ is at hand, the topological equivalence between the two can be determined by sorting and matching the types and lengths of the chains in the decompositions in $O(n \log n)$ time.

It is possible to extend this algorithm to deciding isomorphisms among more complex sparse crossbars. However, the time complexity of such extensions can get out of hand as the components deviate substantially from simple open and close chains in the decompositions. We leave a closer examination of such extensions to the reader as we move to another interesting subject on sparse crossbars.
2.5.2 Density of Sparse Crossbar Concentrators

The equivalence relation among sparse crossbars raises the following fundamental question: For any fixed positive integers \(n, m \leq n\), what is the density of \((n, m)\)-sparse crossbar concentrators among all \(2^{nm}\) \((n, m)\)-sparse crossbars? That is, if \(\text{XC}(n, m)\) denotes the set all \((n, m)\)-sparse crossbar concentrators, then, what is

\[
d_{\text{XC}(n, m)} = \lim_{n \to \infty} \frac{|\text{XC}(n, m)|}{2^{nm}}.
\]

It is easy to see that when \(m = 1\), there is only one sparse crossbar concentrator for all \(n \geq 1\), and hence

\[
d_{\text{C}(n, 1)} = \lim_{n \to \infty} 2^{-n} = 0.
\]

For other values of \(m\), the problem is not as trivial. For example, suppose \(m = n\). In this case, we are dealing with square matrices, and to count all \(n \times n\) matrices in which there is a matching of each column with a distinct row, we must have an actual algorithm to construct all such matrices. For \(n = 2\), it is easy to see that of the sixteen \(2 \times 2\) binary matrices, exactly seven correspond to sparse crossbar concentrators. For \(n = 3\), we can count them by examining the structure of \(3 \times 3\) matrices that correspond to sparse crossbar \((3, 3)\)-concentrators. As a first step, consider those \(3 \times 3\) matrices in which the first row contains a single “1”. For this matrix to correspond to a concentrator, it must have a “1” in its second row which is located in a different column than the one which contains the 1 in the first row. Likewise, the third row must contain a “1” in a column other than the columns in which the first and second rows had their “1”-s. With this observation, it is readily seen that, when the first row is 001, the structure of the concentrator matrices must conform to one of the five patterns, where the “-” entries indicate that the entry can be 0 or 1.

\[
\begin{bmatrix}
0 & 0 & 1 \\
0 & 1 & - \\
1 & - & - \\
\end{bmatrix},
\begin{bmatrix}
0 & 0 & 1 \\
1 & 0 & - \\
- & 1 & - \\
\end{bmatrix},
\begin{bmatrix}
0 & 0 & 1 \\
1 & 1 & - \\
1 & 0 & - \\
\end{bmatrix},
\begin{bmatrix}
0 & 0 & 1 \\
0 & 0 & 1 \\
1 & 1 & - \\
\end{bmatrix},
\begin{bmatrix}
0 & 0 & 1 \\
1 & 1 & - \\
1 & 1 & - \\
\end{bmatrix}.
\]
Noting that the “−” entries can be replaced with 0 or 1, we find that there exist 28 $3 \times 3$ concentrators whose first row is 001. Likewise, each of the $3 \times 3$ matrices whose first row contains 010 or 100 contribute 28 to the total count. The next group of $3 \times 3$ matrices which correspond to concentrators are those whose first row contains exactly two “1”s. Consider those matrices whose first row contains 011. It can be verified that if the second row contains two “1”s, and the locations of these “1”s do not coincide with the columns in the first row then the third row can contain any pattern of “0”s and “1”s, except 000. In case that both “1”s are in the same columns as those in the first row, then the third row must contain a ‘1’ in the first column, and the other entries can be 0 or 1. Analyzing the other cases similarly, we have the following patterns of $3 \times 3$ matrices when the first row is 011, and where $x$ entries indicate that the third row can contain any pattern of “0”s and “1”s except all “0”s.

\[
\begin{bmatrix}
0 & 1 & 1 \\
0 & 0 & 1 \\
1 & - & - \\
\end{bmatrix},
\begin{bmatrix}
0 & 1 & 1 \\
0 & 1 & 0 \\
1 & - & - \\
\end{bmatrix},
\begin{bmatrix}
0 & 1 & 1 \\
1 & 0 & 0 \\
1 & - & - \\
\end{bmatrix},
\begin{bmatrix}
0 & 1 & 1 \\
1 & 0 & 1 \\
- & 0 & 1 \\
\end{bmatrix},
\begin{bmatrix}
0 & 1 & 1 \\
1 & 0 & 0 \\
- & 1 & 0 \\
\end{bmatrix},
\begin{bmatrix}
0 & 1 & 1 \\
1 & 0 & 1 \\
x & x & x \\
\end{bmatrix},
\begin{bmatrix}
0 & 1 & 1 \\
1 & 1 & 0 \\
x & x & x \\
\end{bmatrix},
\begin{bmatrix}
0 & 1 & 1 \\
1 & 1 & 1 \\
x & x & x \\
\end{bmatrix}.
\]

The first three patterns contribute 4 concentrators each, the next three contributes 2 concentrators each, and the last three contributes 7 concentrators each, resulting in a total of 39 concentrators. Similarly, the other two $3 \times 3$ matrices whose first row contains 101 or 110 will each contribute 39 to the total count. Finally, the following patterns of matrices whose first row contains 111 must be taken into account, where again the $x$ entries can be any pattern of “0”s and “1”s except all “0”s, and the “−” entries can be 0 or 1.

\[
\begin{bmatrix}
1 & 1 & 1 \\
0 & 0 & 1 \\
x & x & - \\
\end{bmatrix},
\begin{bmatrix}
1 & 1 & 1 \\
0 & 1 & 0 \\
x & - & x \\
\end{bmatrix},
\begin{bmatrix}
1 & 1 & 1 \\
1 & 0 & 0 \\
- & x & x \\
\end{bmatrix}.
\]

Adding these matrices up we find that they contribute $3 \times 6 + 4 \times 7 = 46$ concentrators to the total count. Hence, the total number of sparse crossbar $(3, 3)$-concentrators is given by

$$3 \times 28 + 3 \times 39 + 46 = 247.$$ 

By extending these counting arguments, it can be shown that the total number of sparse crossbar $(4, 4)$-concentrators is given by $37811$. These numbers show that $d_{C(2,2)} = 0.437$, $d_{C(3,3)} = 0.482$, $d_{C(4,4)} = 0.576$.

They further suggest that $d_{C(n,n)} \rightarrow 1$ as $n \rightarrow \infty$, but the most we can show that $d_{C(n,n)} \geq 0.288$ as $n \rightarrow \infty$.

**Theorem 2.33** The density of sparse crossbar $(n, n)$-concentrators is at least $0.288$ as $n \rightarrow \infty$.

**Proof:** The main idea is to construct as many $n \times n$ binary matrices corresponding to sparse crossbar concentrators as possible. One way to accomplish is to fix a “1” in each row no two of which are in the same column. This can be insured if we iteratively avoid entering all zero tuples into the rows. Hence, the first row can be assigned any of the $2^n - 1$ non-zero $n$ binary tuples, the second row (except its first column) can be assigned any of $2^{n-1} - 1$ non-zero $(n - 1)$ binary tuples, the third row (except its first two columns) can be assigned any of $2^{n-2} - 1$ non-zero $(n - 2)$ binary tuples, and so on. This proves that the number of sparse crossbar $(n, n)$-concentrators is not less than

$$2^n (\prod_{i=0}^{n-1} 1 - 2^{-i-n}) = 2^n (\prod_{i=1}^{n} 1 - 2^{-i})$$
where the first term in the first product accounts for the number of ways the unused columns in the \( i \)th row can be set for each \( n-i \)-tuple entered into the remaining columns. Now, dividing the expression by \( 2^{n^2} \), we have the lower bound

\[
d_{C(n,n)} = \lim_{n \to \infty} \prod_{i=1}^{n} (1 - 2^{-i}). \quad (2.41)
\]

Using the Euler expansion [?]

\[
\prod_{i=1}^{\infty} (1 - q^i) = 1 - (q^1 + q^2) + (q^5 + q^7) - (q^{12} + q^{15}) + \ldots,
\]

with \( q = 1/2 \), the indices being alternately \( n(3n \pm 1)/2 \), we find that

\[
d_{C(n,m)} \geq 1 - (1/2+1/4)+(1/32+1/128)-(1/1024+1/32768) \geq 0.288
\]

as claimed. ||

The argument of the proof provides an algorithm to construct sparse crossbar \((n, n)\)-concentrators. Here we illustrate the construction for \( n = 2 \). We start out with the first row, and write out all \( 2 \times 2 \) matrices with the first row containing any \( 2 \)-tuple but 00.

\[
\begin{pmatrix}
0 & 1 \\
- & -
\end{pmatrix},
\begin{pmatrix}
1 & 0 \\
- & -
\end{pmatrix},
\begin{pmatrix}
1 & 1 \\
- & -
\end{pmatrix},
\]

We next fix the column in the next row which we must leave out to ensure that the “1”s in the two rows will appear in two columns. This depends on the location of the “1” entry in the first row, and in the case that both columns in the first row are “1”s there is a choice. The unused column in the second row can be 0 or 1, and hence, we get two matrices from each of the matrices above as follows:

\[
\begin{pmatrix}
0 & 1 \\
1 & 0,1
\end{pmatrix},
\begin{pmatrix}
1 & 0 \\
0,1 & 1
\end{pmatrix},
\begin{pmatrix}
1 & 1 \\
1 & 0,1
\end{pmatrix}.
\]

We note that, in this simple case, the lower bound count 6 of the 7 sparse crossbar \((2, 2)\)-concentrators.
So far, we only examined the most trivial case about the density of sparse crossbar concentrators. Here we generalize it to other values of \( m \), and give two lower and two upper bounds.

### A. Lower Bounds

To find a lower bound on \( d_{C(n,m)} \), we count the number of \( m \times n \) matrices that correspond to an \( (n,m) \)-concentrator, and can be generated from a fat-and-slim concentrator using Corollary 2.11.

First assume that \( m \leq n - m \). This makes the fat region of the concentrator at least as large as the slim region. Suppose we choose any \( i \) columns from the fat region and any \( i \) columns from the slim region, and swap some entries between the first columns selected from each group, then between second columns selected from each group, and so on. For each pair of columns, there can be at most \( m - 1 \) swaps, yielding \( 2^{m-1} - 1 \) different arrangements of the two columns (we exclude the case of no swaps). This gives \( (2^{m-1} - 1)^i \) different concentrators for each selection of the groups. Summing this up for all the possible selections, we get

\[
|C(n, m)| \geq \sum_{i=0}^{m} \binom{m}{i} \binom{n-m}{i} (2^{m-1} - 1)^i.
\]

Now, suppose we add \( k \) columns to the fat region, and remove \( k \) columns from the slim region. For each \( k \), we will get a different concentrator since the total number of crosspoints are changed. We can use the above generation process on the concentrators obtained in this way, which modifies the bound as

\[
|C(n, m)| \geq \sum_{k=0}^{m} \sum_{i=0}^{m-k} \binom{m-k}{i} \binom{n-m+k}{i} (2^{m-1} - 1)^i, \tag{2.42}
\]

\[
= \sum_{i=0}^{m} \sum_{k=0}^{m-i} \binom{m-i-(k-i)}{i} \binom{n-m+k}{i} (2^{m-1} - 1)^i, \tag{2.43}
\]

\[
\geq \sum_{i=0}^{m} \sum_{k=0}^{m-i} \binom{m-i-k}{i} \binom{n-m+k}{i} (2^{m-1} - 1)^i,
\]

\[
\geq \sum_{i=0}^{m} \binom{n-i+1}{2i} (2^{m-1} - 1)^i. \tag{2.43}
\]
We can use the same counting technique for the \( m > n - m \) case. In this case, the summation limits will be different. Also, increasing the fat region will eventually bring the problem to the previous case, so there will be two summation terms. The lower bound in this case is thus given by

\[
|C(n, m)| \geq \sum_{k=0}^{m-n/2} \sum_{i=0}^{n-m+k} \binom{m-k}{i} \binom{n-m+k}{i} (2^{m-1} - 1)^i + \sum_{k=1}^{n/2-n-k} \sum_{i=0}^{n/2-k} \binom{n/2-k}{i} \binom{n/2+k}{i} (2^{m-1} - 1)^i
\]

This bound can be simplified as in the previous case, and dividing (2.42) and (2.44) by \( 2^{nm} \) yields a lower bound on the density of sparse crossbar concentrators, i.e.,

\[
d_C(n, m) \geq \lim_{n \to \infty} \frac{|C(n, m)|}{2^{nm}}.
\]

However, it is not difficult to see that, in both cases, the lower bound on the density tends to 0 as \( n \to \infty \).

This lower bound can be improved by generalizing the lower bound technique for the \( m = n \) case. Again, the key idea is to ensure that each input has a distinct output to be matched with. This is guaranteed if we enter a “1” in a different column in each row as we prove in the next theorem.

**Theorem 2.34**: There exist at least

\[
\left( \sum_{i=n-m+1}^{n} \binom{n}{i} \right) \times \left( \prod_{i=1}^{m-1} 2^i (2^{m-i} - 1) \right)
\]

sparse crossbar concentrators.

**Proof**: We know from Nakamura-Masson lower bound, i.e., Theorem 2.4, that every row in the incidence matrix of a sparse crossbar concentrator must have at least \( n - m + 1 \) “1”s. Hence, the first row can be fixed in any one of

\[
\sum_{i=n-m+1}^{n} \binom{n}{i}
\]
ways. Once the first row is fixed this way, we fill some $n - m$ columns which have “1” entries in the first row with “1”s, and mark a “1” among the remaining “1”s in the first row. This insures that the input which corresponds to the column with the marked “1” can be matched with an output. This process can be iterated by allowing the remaining $m - 1$ columns in the second row to assume any one of $2^{m-1} - 1$ patterns of “1”s and “0”s (we only need to exclude the all “0” pattern). Furthermore, the entry at the intersection of the second row and the column in which the 1 has been marked can be either 0 or 1. Hence, the unspecified $m$ entries in the second row can be fixed in any one of $2(2^{m-1} - 1)$ ways. Iterating this argument for the next row, and so on, we see that the total number of sparse crossbar $(n, m)$-concentrators is not less than the expression given in 2.45.

**A. Upper Bounds:**

The most obvious upper bound can be derived using Nakamura-Masson lower bound. The theorem states that every sparse crossbar $(n, m)$-concentrator must have at least $(n - m + 1) m$ crosspoints. It follows that all $m \times n$ matrices with fewer than $(n - m + 1) m$ “1”s should correspond to bipartite graphs which are not concentrators. Hence,

$$d_{C(n,m)} \leq \lim_{n \to \infty} \frac{2^{nm} - \sum_{i=0}^{(n-m+1)m-1} \binom{nm}{i}}{2^{nm}},$$

(2.46)

or

$$d_{C(n,m)} \leq \lim_{n \to \infty} \frac{\sum_{i=(n-m+1)m}^{nm} \binom{nm}{i}}{2^{nm}}.$$  (2.47)

Using Moon’s inequality[?],

$$\sum_{i>r/2+\lambda} \binom{r}{i} < 2^r e^{-2\lambda^2/r},$$

in the interval $0 \leq \lambda \leq r/2$, with $r = nm$, $\lambda = nm/2 - m(m-1) - \epsilon$, where $\epsilon$ is an infinitely small positive number, yields the upper bound

$$d_{C(n,m)} \leq \lim_{n \to \infty} e^{-2((nm/2-m(m-1))^2/nm)},$$

(2.48)
or

\[ d_{C(n,m)} \leq \lim_{n \to \infty} e^{-2m((n/2-(m-1))^2/n} \tag{2.49} \]

in the interval \( 1 \leq m \leq n/2 + 1 \). It follows that

**Corollary 2.35:**

\[ d_{C(n,n/2)} \leq 1/e \approx 0.37. \tag{2.50} \]

We can derive another upper bound on \( d_{C(n,m)} \) in the same interval by lower bounding the number of bipartite graphs which are not concentrators as above. This time, we count those bipartite graphs in any row of which there are \( m \) or more zeros. Again, by Theorem ??, these correspond to bipartite graphs which are not concentrators.

Let \( R_k \) denote the set of \( m \times n \) matrices with \( m \) or more zeros in the \( k \)th row. Let \( |R_k| \) denote the number of elements in \( R_k \). Then,

\[
|R_{j_1}| = \sum_{i=m}^{n} \binom{n}{i} 2^{n(m-1)},
\]

\[
|R_{j_1} \cap R_{j_2}| = \left( \sum_{i=m}^{n} \binom{n}{i} \right)^2 2^{n(m-2)},
\]

\[
\vdots
\]

\[
|R_{j_1} \cap \cdots \cap R_{j_m}| = \left( \sum_{i=m}^{n} \binom{n}{i} \right)^m,
\]

for any distinct indices \( j_1, \ldots, j_m \).

We can use these equalities to derive a lower bound on the number, \( x_{C(n,m)} \), of bipartite graphs which are not concentrators with the inclusion exclusion principle.

\[
x_{C(n,m)} \geq |R_1 \cup R_2 \cup \cdots \cup R_m| = \sum_i |R_i| - \sum_{i,j} |R_i \cap R_j| + \cdots + (-1)^{m+1} |R_1 \cap \cdots \cap R_m|
\]

\[
= \sum_{j=1}^{m} \binom{m}{j} (-1)^{j+1} \left( \sum_{i=m}^{n} \binom{n}{i} \right)^j 2^{n(m-j)}
\]

\[
= -2^{nm} \sum_{j=1}^{m} \binom{m}{j} \left( \sum_{i=m}^{n} \binom{n}{i} (-2^{-n}) \right)^j.
\]
Using the equality
\[
\sum_{j=1}^{m} \binom{m}{j} x^j = (x + 1)^m - 1,
\]
we get
\[
x_{C(n,m)} \geq -2^{nm} \left( \sum_{i=m}^{n} \binom{n}{i} (-2^{-i} + 1)^m - 1 \right),
\]
\[
= 2^{nm} - (2^n - \sum_{i=m}^{n} \binom{n}{i})^m
\]
\[
= 2^{nm} - \left( \sum_{i=0}^{m-1} \binom{n}{i} \right)^m.
\]
Since \( d_{C(n,m)} = 1 - \lim_{n \to \infty} (x_{C(n,m)}/2^{nm}) \), we get
\[
d_{C(n,m)} \leq \lim_{n \to \infty} \left( \frac{1}{2^n} \sum_{i=0}^{m-1} \binom{n}{i} \right)^m.
\]
(2.51)

Rewriting the inequality as
\[
d_{C(n,m)} \leq \lim_{n \to \infty} \left( \frac{1}{2^n} \sum_{i=n-m+1}^{n} \binom{n}{i} \right)^m.
\]
(2.52)

and once again, using Moon’s inequality with \( \lambda = (n-m+1) - n/2 - \epsilon \), where \( \epsilon \) is an arbitrarily small positive integer, we obtain the same upper bound
\[
d_{C(n,m)} \leq \lim_{n \to \infty} e^{-2m(n/2-m+1)^2/n}.
\]
(2.53)
in the interval \( 1 \leq m \leq n/2 + 1 \).

As we will see in Section 6, if the second bound is computed directly using the expression in (2.52) it provides a tighter upper bound on \( d_{C(n,m)} \). This bound can be improved by considering larger values of \( k \) in Theorem ???. However, the counting gets more complicated, and lest we count the non-concentrator graphs due to the higher order terms precisely, their contribution is negligible on \( x_{C(n,m)} \).
2.5.3 Concentration Power of Sparse Crossbars

So far we have been concerned with the characterizations of sparse crossbars that have the ability to concentrate any given subset of their inputs up to a capacity $c \leq m$. The notion of capacity provides a tangible measure of the concentration capability of sparse crossbars, but it does not differentiate between sparse crossbars with vastly different sets of transversals (unordered matchings). Indeed, two sparse crossbars can have the same capacity, but one may realize many more transversals than the other. As an example, consider two $(2m,m)$-sparse crossbars $G_1$ and $G_2$ where $G_1$ has all of its $2m$ crosspoints placed in a single row, and $G_2$ has the same crosspoints plus another $2m - 2$ crosspoints placed in another row. Obviously, both have a capacity of 1, but $G_2$ can realize a total of $2m + \binom{2m}{2} - 1$ transversals whereas $G_1$ can realize only $2m$ transversals. If this sounds surprising, let us also mention that a sparse crossbar can even realize more transversals than another sparse crossbar with a larger capacity (See Problem 2.57).

The foregoing discussion underscores the need for a stronger measure of performance in determining how closely a given sparse crossbar approximates a concentrator, and one such measure, which we will call the concentration power, can be obtained by comparing the number of transversals realized by a sparse crossbar to the total number of transversals possible. More precisely, let $G$ be an $(n,m)$-sparse crossbar, and let $T_G$ be the set of $k$-transversals, $1 \leq k \leq m$, realized by $G$. The concentration power of $G$, denoted $\Xi_G$, is given by the ratio

$$\Xi_G = \frac{|T_G|}{\sum_{k=0}^{m} \binom{n}{k}}.$$  

It is obvious that $G$ is an $(n,m)$-concentrator if and only if $\Xi_G = 1$. In general, short of proving that a sparse crossbar is a $(n,m)$-concentrator (which would then imply that its concentration power is 1), computing its concentration power requires the enumeration of all $k$-transversals, $1 \leq k \leq m$, it can realize. On the other hand, in some cases, we can use a watered down version of the concentration power notion, which we will call the $m$th degree concentration power, to establish that a given sparse crossbar is an $(n,m)$-concentrator.
Let us define the $m$th degree concentration power of an $(n,m)$-sparse crossbar as

$$\Xi_{G,m} = \frac{|T_{G,m}|}{\binom{n}{m}}.$$ 

It is easy to see that if $\Xi_G = 1$ then we must have $\Xi_{G,m} = 1$. What is also true is that if $\Xi_{G,m} = 1$ then $\Xi_G = 1$. This is because if a sparse crossbar realizes all $m$-transversals then, by Proposition 2.1, it also realizes all $k$-transversals, $1 \leq k \leq m-1$, and hence its concentration power must be 1.

Thus, to show that $G$ is an $(n,m)$-concentrator, it suffices to show that its $m$th degree concentration power is 1. This in turn amounts to counting all of its $m$-transversals which remains largely an unsolved problem for an arbitrary sparse crossbar. A related counting problem, namely, the enumeration of 1-factors (ordered matchings) in bipartite graphs has been extensively studied, and it was shown by Valiant that determining the exact number of 1-factors of an arbitrary bipartite graph is NP-hard [BR91].

This latter number is well-known in combinatorial matrix theory, and is called the permanent. More precisely, let $A = [a_{ij}]$ be an $m \times n$ matrix, where $m \leq n$. The permanent of $A$ is defined by

$$\text{Per}(A) = \sum a_{1,x_1}a_{2,x_2}\ldots a_{m,x_m}$$

where the summation extends over all ordered tuples $x_1x_2\ldots x_m$ where $x_1 \neq x_2 \neq \ldots \neq x_m \in \{1,2,\ldots,n\}$. Now, if $G$ is an $n$-input, and $m$-output sparse crossbar with the adjacency matrix, $A_G = [a_{ij}]$, then it is seen that the number of 1-factors (ordered $m$-matchings) between the $m$ outputs and $n$ inputs of $G$ is given by $\text{Per}(A_G)$. For example, the permanent of the sparse crossbar shown in Figure 2.31(a) is computed by summing all ordered products $a_{1,x_1}a_{2,x_2}a_{3,x_3}a_{4,x_4}$ in the matrix

$$A_G = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}.$$ 

In this case, there are exactly nine non-zero such products so that $G$ has nine 1-factors. Two of these are shown in Figure 2.31(b). In
this case, it turns out that each 1-factor corresponds to a distinct 4-
transversal so that \( \mathcal{G} \) has nine 4-transversals as well. Hence, by the 
above definition, the 4th degree concentration power of \( \mathcal{G} \) is \( 9/15 \), 
and since this is \( < 1 \), \( \mathcal{G} \) is not a (6, 4)-concentrator.

In general, the direct computation of the permanent of an adja-
cency matrix by the above summation formula requires summing 
\( n(n - 1) \ldots (n - m + 1) \) ordered products. While this computation 
cannot be circumvented to get the exact value of the permanent of a 
matrix, a vast literature exists on lower and upper bounds on perma-
nents of matrices, in part, due to the celebrated Van Der Waerden’s 
conjecture on the permanent of doubly stochastic matrices\(^{10}\).

One of these bounds, due to Ostrand, ties the permanent of an ad-
jacency matrix to the number of “1”’s in its rows.

**Theorem 2.36** [Ostrand [BR91]] *Let* \( A = [a_{ij}] \) *be an* \( m \times n \) *binary 
matrix, where* \( m \leq n \), *and let* \( r_1, r_2, \ldots, r_m \) *denote the sums of the 
entries in rows* \( 1, 2, \ldots, m \), *in that order. Suppose that the rows of 
\( A \) *have been arranged so that* \( r_1 \leq r_2 \leq \ldots \leq r_m \). *Then*

\[
\text{Per}(A) \geq \prod_{i=1}^{m} \text{Max}\{1, r_i - i + 1\}.
\]

**Proof:** It is left as an exercise. \( || \)

This fact facilitates a counting argument to prove that the full cross-
bar is a \((n, m)\)-concentrator. This can be seen by noting that, for an 
\( m \times n \) full crossbar, the above inequality reduces to

\[
\text{Per}(A) \geq \prod_{i=1}^{m} n - i + 1 = n(n - 1) \ldots (n - m + 1).
\]

Thus, the number of 1-factors of an \( m \times n \) full crossbar is \( \geq n(n - 1) \ldots (n - m + 1) \). Moreover, since each transversal can be identified

---

\(^{10}\) This conjecture which was proved independently by G. P. Egorychev and D. I. 
Falikman states that the permanent of a doubly stochastic matrix, an \( n \times n \) matrix 
in which each row and each column sum is 1, is \( \leq n!/n^n \), with equality holding 
only for the \( n \times n \) matrix of all 1/n entries (See Section 7.4 in [JR86,BR91]).
Figure 2.31: A sparse crossbar and two of its nine 1-factors.
with at most \( m! \) 1-factors, this inequality implies that the number of \( m \)-transversals in an \( m \times n \) full crossbar must be no less than

\[
n(n - 1) \ldots (n - m + 1)/m! = \binom{n}{m}
\]

which is the requisite number of matchings for an \((n, m)\)-concentrator.

We can use the same counting approach to prove that a fat-and-slim crossbar is an \((n, m)\)-concentrator. In this case, we need to identify the 1-factors with \( m \)-transversals more accurately. Recall that an \((n, m)\)-fat-and-slim crossbar \( G \) is obtained by dividing the \( n \) inputs into two sets, \( I_1 \) and \( I_2 \), where \( |I_1| = n - m \) and \( |I_2| = m \), and connecting each input in \( I_1 \) to all \( m \) outputs, and connecting each input in \( I_2 \) to a distinct output. In matrix terms, this amounts to an adjacency matrix \( A_G = [F_{m \times n-m} | I_{m \times m}] \) where \( F_{m \times n-m} \) is an \( m \times n - m \) all \( 1 \)'s matrix and \( I_{m \times m} \) is an \( m \times m \) identity matrix.

To compute the number of \( m \)-transversals in \( G \), all we need to do is count the number of \( m \)-transversals associated with each of the \( m \times m \) submatrices of \( G \). One way to do this is to form these submatrices by choosing \( m - i \) columns from \( I_1 \) and \( i \) columns from \( I_2 \). Since each of the \( m - i \) columns in \( I_1 \) has all \( 1 \)'s entries, and each of the \( i \) columns in \( I_2 \) has a single \( 1 \) entry, the number of 1-factors associated with all \( m \times m \) submatrices obtained by concatenating \( m - i \) columns in \( I_1 \) with \( i \) columns in \( I_2 \) is given by \( \binom{n-m}{m-i} \binom{m}{i} (m-i)! \), and the number of \( m \)-transversals is obtained by dividing this number by \( (m - i)! \) since all 1-factors obtained by permuting the columns in \( I_1 \) and only those columns are associated with the same \( m \)-transversal.

Adding these numbers for \( i, 1 \leq i \leq m \), the total number of \( m \)-transversals realized by \( G \) is given by

\[
\sum_{i=0}^{m} \left( \binom{n-m}{m-i} \binom{m}{i} \right) = \binom{n}{m}
\]

by Vandermonde’s convolution formula. Hence, \( G \) realizes all \( \binom{n}{m} \) \( m \)-transversals and qualifies to be an \((n, m)\)-concentrator.
While these examples demonstrate the relation between permanents of non-square matrices and concentration power of sparse crossbar concentrators, the known bounds on permanents in general do not yield tight estimates of the concentration power of a sparse crossbar, and further research is needed to develop sharp bounds on the number of transversals in bipartite graphs.

2.6 BIBLIOGRAPHICAL NOTES

The word "concentrator" appears to have entered the interconnection network literature in the fifties through patents and technical reports on telephone switching. The earliest patent the author could find and in which concentrators are explicitly mentioned dates back to 1955 [ER55]. Given that the telephone was invented only 120 years ago, and that automated telephone toll switching systems have been in existence only since 1920’s, this is not too surprising.

Concentrators (frequently called line concentrators in telephone switching) are used to multiplex subscriber loops within small patches of residential areas to voice trunks that are then connected to central switching offices. AT&T’s SLC-40, SLC-96 and SLC-2000 switches are examples of such concentrators. The SLC-96 switch contains four multiplexers each of which combines twenty-four 64 Kbits/s channels onto a 1.544 Mbits/s channel—a T1 line [Can81]. The topological structure of these concentrators fits our model of a concentrator in that some 96 inputs are concentrated to some 4 outputs. Nonetheless, these concentrators are designed so as to transmit all 96 voice channels over four T1 lines without any blocking. This is very much like the space and time division scheme we outlined in Section 1, where some 96 voice channels are divided to four groups of 24 channels, and each voice channel in each group is sampled once every 125 µ secs and transmitted within a fixed time slot on a T1 line using an 8-bit pulse coded modulation (PCM) representation. Time division multiplexing and other practical aspects of concentrators are covered in great detail in [TT89].

Sparse concentrators described in this chapter are space division multiplexers. They were first treated formally in [Pin73], and sub-
sequently in [Mas77,MGN79,NM82,GHR88] and [OH94]. Binomial concentrators were introduced in [Mas77] and further extended in [NM82]. Fat-and-slim crossbars were described in [OH94]; Balanced sparse crossbars were first introduced in [OH94], and treated formally in [GO96b]. Lower bounds on the crosspoint complexity of sparse crossbar concentrators were derived in [NM82,OH94]. The results on bounded capacity sparse crossbar concentrators were reported in [GO96a].

Graph theoretical facts, in particular Hall’s and König’s theorems, can be found in any standard text on graph theory, see for example, [Ber85]. Mirsky’s monograph provides an indepth coverage of all these theorems and many of their variations from a transversal theory point of view [Mir71]. Brualdi and Ryser [BR91] offer a combinatorial matrix theory view of bipartite graphs, and also provides a comprehensive coverage on permanents.

REFERENCES


2.7 EXERCISES

Problem 2.1 [25] The multiplexing transparency of a space-and-time division concentrator is defined as the ratio

\[
\frac{\text{number of feasible channel patterns}}{\text{total number of channel patterns}}
\]
(a) Show that the multiplexing transparency of a space-and-time division concentrator with \( r \) identical multiplexers, \( x \) channels and total capacity \( \gamma \) is given by

\[
\left( \frac{x}{r} \right)^r \left( \frac{x}{\gamma} \right).
\]

(b) Show that the multiplexing transparency of a space-and-time multiplexer is a convex decreasing function of the number of multiplexers.

**Problem 2.2** [15] Prove Propositions 2.1, 2.2, and 2.3.

**Problem 2.3** [15]
(a) Verify the relations among various concentrators described in Figure 2.4.
(b) Prove that the set of strong concentrators include \((n, m, c)\)-strong concentrators, with capacity strictly less than \( m \) that are also \((n, m)\)-concentrators.

**Problem 2.4** [25]
(a) Show that in any strong \((n, m, c)\)-concentrator, there exist sets of outputs, \( Y_1 \subseteq Y_2 \subseteq \ldots \subseteq Y_c \) such that \( Y_k \) is a subset of every \( k \) inputs.
(b) Suppose that in a graph with \( n \) inputs and \( m \) outputs, every subset of \( k \) inputs has a matching set, and the matching set of every \( k \) inputs is contained in a matching set of every \( k + 1 \) inputs. Show that this graph is not necessarily a strong concentrator.

**Problem 2.5** [25]
(a) Show that the following network is an \((8, 4)\)-strong concentrator, but not an \((8, 4)\)-superconcentrator.
(b) Can this construction be extended to \( 2n \) inputs and \( n \) outputs, \( n \geq 1 \)? Explain.
Problem 2.6 [10] Show that every sparse crossbar \((n, m, c)\)-superconcentrator, \(1 \leq c \leq m - 1\), is also an \((n, m)\)-superconcentrator.

Problem 2.7 [10] Show that the bipartite version of the pruned strong concentrator described in Section 2.2.2 has the minimum crosspoint complexity possible for a bipartite \((4, 3)\)-strong concentrator.

Problem 2.8 [10]
(a) Construct a sparse crossbar \((5, 3)\)-concentrator with 9 cross-points.
(b) Construct a sparse crossbar \((9, 5, 2)\)-concentrator with 13 cross-points.

Problem 2.9 [20, Nakamura-Masson]
(a) Let \(M\) be an \(m \times n\) binary matrix, and let \(r_i\) and \(s_j\) denote the number of “1”s in the \(i\)th row and \(j\)th column, respectively. I. Tomescu, on page 175 of his book *Introduction to Combinatorics*, attributes the following inequality to Khintchine:

\[
\sum_{1 \leq i \leq m} r_i^2 + \sum_{1 \leq j \leq n} s_j^2 \leq \sigma \left( t + \frac{\sigma}{t} \right)
\]

where \(t = \text{Max}\{n, m\}\), and \(\sigma\) is the total number of “1”s in the matrix. Use Khintchine’s inequality to show that the crosspoint complexity, \(\kappa(n, m)\) of a sparse crossbar \((n, m)\)-concentrator satisfies the inequality

\[
\kappa^2(n, m) + n^2 \kappa(n, m) - m(n + m)(n - m + 1)^2 \geq 0.
\]

(b) Use this inequality to obtain a lower bound on \(\kappa(n, m)\) and compare it with the lower bound of Theorem 2.4.

Problem 2.10 [10]
(a) Show that there exists exactly one bipartite realization of an \((n, m)\)-superconcentrator.
(b) Show that every sparse crossbar \((n, m)\)-superconcentrator must have \(nm\) crosspoints.
Problem 2.11  [10]  
Show that the following graph is a (4,3,2)-superconcentrator, but not a (4,3)-superconcentrator.

Problem 2.12  [25] Show that there exist exactly

\[ m! \prod_{i=1}^{m} \binom{n}{n-i+1} \]

sparse crossbar strong \((n,m)\)-concentrators with a minimum crosspoint complexity.

Problem 2.13  [10] Determine the capacity of the sparse crossbar shown below.

Problem 2.14  [20] A fixed ratio concentrator with factor \( r \) is an \((nr, mr, cr)\)-concentrator, where \( c \leq m \leq n \), and \( r \) are positive integers. Show that the crosspoint-count of a sparse crossbar fixed ratio
concentrator with factor $r$ is greater than $nr$, where $x$ satisfies the inequality,

$$x \ln \frac{mr}{cr - 1} + x^2 \left( \frac{1}{cr - 1} - \frac{1}{mr} \right) \geq \ln \frac{nr}{cr - 1}. $$

**Problem 2.15** [20, Nakamura-Masson] Use Theorem 2.17 to show that any sparse crossbar $(16, 7, 6)$-concentrator requires 46 crosspoints. Construct an $(16, 7, 6)$-concentrator whose crosspoint complexity matches this bound as closely as possible (Hint: 48 crosspoints are sufficient.)

**Problem 2.16** [20, Nakamura-Masson]
(a) Prove that the sparse crossbar network shown below is a $(20, 6, 5)$-concentrator.
(b) Show that its crosspoint complexity matches Nakamura-Masson’s lower bound.
(c) Is this a binomial network?
Problem 2.17 [20]
(a) Show that Nakamura-Masson’s lower bound on the crosspoint complexity is given by \( nx_s \) where \( x_s \) is the solution of the identity
\[
\prod_{i=0}^{m-c} (1 - \frac{x_s}{m-i}) = \frac{c-1}{n}.
\]
(b) Use the above relation to derive lower and upper bounds on Nakamura-Masson’s lower bound. (Hint: Use Weistrass inequalities, \( 1 - \sum_{k=1}^{n} x_k \leq \prod_{k=1}^{n} (1 - x_k) \leq (1 + \sum_{k=1}^{n} x_k)^{-1}, 0 < x_k < 1 \).
(c) Show that Oruç-Huang’s lower bound lies between the lower and upper bounds on Nakamura-Masson’s lower bound.

Problem 2.18 [10] Prove that the capacity of a sparse crossbar cannot exceed the minimum number of its inputs and outputs that contain between them all the crosspoints. (Hint: Use König’s theorem.)

Problem 2.19 [25]
(a) Prove Theorem 2.9.
(b) Extend the construction in this theorem to include the case \( k \) does not divide \( m \).


Problem 2.21 [15]
Prove that the capacity of the binomial \( \binom{m}{v} \)-network does not exceed \( v + 2 \), for \( v \geq 2 \).
Problem 2.22 [30] Determine the capacity of each of the following sparse crossbars.

![Crossbar Diagrams](image)


Problem 2.24 [20] Balance each of the sparse crossbars in Problem 2.22 so that each input has an out-degree of 3 ± 1.

Problem 2.25 [20] Let $G$ be an $(n,m)$-sparse crossbar whose inputs are labeled $1, 2, \ldots, n$, from left to right, and whose outputs are labeled $1, 2, \ldots, m$ from top to bottom. Suppose that $G$ has a crosspoint at all intersections of inputs and outputs except when $i < m - j + 1$ or $i > n - j + 1$, $i = 1, 2, \ldots, m$. Show that $G$ is a full capacity concentrator. Is this an optimal concentrator?

Problem 2.26 [20]
(a) Prove that the $(2m, m)$-sparse crossbar shown below, where the shaded regions indicate the crosspoint locations is an optimal full capacity concentrator.

(b) What is the capacity of the sparse crossbar obtained by reflecting the crosspoints in the second half along its main diagonal? Explain.
Problem 2.27 \[20\]
(a) Prove that the \((3m, m)\)-sparse crossbar shown below, where the shaded regions indicate the crosspoint locations is an optimal full capacity concentrator.
(b) Can this sparse crossbar concentrator be extended to \(km\) inputs and \(m\) outputs, \(k \geq 3\)? Explain.

Problem 2.28 \[20\]
(a) Prove that the \((2m, m)\)-sparse crossbar shown below, where the shaded regions indicate the crosspoint locations is an optimal full capacity concentrator.
(b) What is the capacity of the sparse crossbar obtained by reflecting the crosspoints in the second half along its main diagonal? Explain.
Problem 2.29  [15] Show how a (12,4)-fat-and-slim crossbar can be balanced to have 3 crosspoints in each column using Theorem 2.10.

Problem 2.30  [30] Give a characterization of optimal sparse crossbar $(n,m)$-concentrators with a constant fanin and a constant fanout.

Problem 2.31  [20] Show that the fraction terms in Eqn. 2.15 can exceed 1 in the region they are defined if $m \geq 2n/3$.

Problem 2.32  [20] Let $G = G_J \oplus G_U$, be a sparse crossbar where $G_1$ is a $m \times (m-\alpha)$ fully connected sparse crossbar, and $G_2$ is an $m \times (\alpha-1)$ upper triangular sparse crossbar. Show how the crosspoints in the columns of $G$ can be balanced using Theorem 2.10.

Problem 2.33  [25] In the proof of Theorem 2.13, we considered two cases for balancing the crosspoints of a $(n,m)$-sparse crossbar.
(a) Show that when $n = 18$ and $m = 14$, the expression in Eqn. 2.18 is $> 0$.
(b) Justify the claim in the second part of the proof by showing that the extra crosspoints in the fat section of the crossbar can indeed be moved over into the triangular and banded sections to have $3 \pm 1$ crosspoints in each column.
(c) Show that $\gamma$ defined in Eqn. 2.18 can be $> 0$ for $\alpha = \lfloor (n-m+1)m/n \rfloor$.

Problem 2.34  [20] Justify the formula in Theorem 2.15 without relying on the combinatorial argument given in the proof of the theorem?
Problem 2.35  [20] Prove that the lower bound expression in Eqn. 2.19 is a convex decreasing function of $c$.

Problem 2.36  [20]
(a) Show that the crosspoint complexity of any sparse crossbar $(n, m, c)$-concentrator is not less than $2n + c - m - 2$ for all $2 \leq c \leq m$.

(b) Determine when this bound is tighter than the lower bound in Eqn 2.19.
(c) Show that this bound is indeed tight when $n = m + 1$.

Problem 2.37  [25]
(a) Prove that the crosspoint complexity of a sparse crossbar $(n, m, c)$-concentrator must be greater than or equal to

\[
\frac{\sum_{i=1}^{c} \binom{m}{i-1}(n - i + 1)}{\sum_{i=1}^{c} \binom{m-1}{i-1}}
\]

(b) Compare this bound with the lower bounds obtained in Section 2.3.

Problem 2.38  [20] Show that the capacity of a sparse crossbar $(n, m, c)$-concentrator with $\kappa > m$ crosspoints must satisfy the following inequality:

\[
c \leq \frac{\kappa(m + 1) - (n + 1)m}{\kappa - m}
\]

Problem 2.39  [25] Give a sparse crossbar $(2^p, 2^p - 1)$-concentrator construction with $1 \leq \text{fanout} \leq 2$ and fanin = 2 for all $p \geq 1$.

Problem 2.40  [15] Show that each of the following sparse crossbars is an optimal $(8,7)$-concentrator.
Problem 2.41  [50] A sparse crossbar is called minimal if deleting any of its crosspoints reduces its capacity by at least 1.
(a) Show that every optimal sparse crossbar \((n, m, c)\)-concentrator is also minimal.
(b) Show that every minimal sparse crossbar \((n, n-1, n-1)\)-concentrator is also optimal.
(c) Prove or disprove that every minimal sparse crossbar \((n, m, c)\)-concentrator, \(c \leq m \leq n\), is also optimal.

Problem 2.42  [40] The \(c\)-concentrator density of a set of \((n,m)\) sparse crossbars, \(F(n,m)\) is defined by

\[
\frac{|F(n,m,c)|}{|F(n,m)|}
\]

where \(F(n,m,c)\) is the set of all \((n,m,c)\)-concentrators in \(F(n,m)\). Show that the \(m/16\)-concentrator density of the set of all \((n,m)\)-sparse crossbars each of whose inputs has fanout = 3 is \(> 0\). That is, show that there exist an \((n,m,m/16)\)-concentrator with \(3n\) crosspoints.
Problem 2.43 [40] Prove that the capacity of the sparse crossbar construction in Figure 2.24 is at least
\[
\min \left( \frac{m}{d^2}, \frac{m}{2d} \times \left( \frac{1}{\sqrt{m}} - \frac{1}{d} + \sqrt{4 + \left( \frac{1}{\sqrt{m}} - \frac{1}{d} \right)^2} \right) \right).
\]

Problem 2.44 [20] Prove Theorem 2.27.

Problem 2.45 [25] Use König’s Theorem to show that the binomial \( \binom{m+2}{m} \)-network is an \( \binom{m+2}{m}, m \)-concentrator.

Problem 2.46 [50] Corollary 2.28 provides a characterization of sparse crossbar \( (n, m) \)-concentrators in terms of an inequality between the degrees of inputs and outputs in subgraphs comprising \( m \) subsets of inputs and the \( m \) outputs. Give a characterization that rely on the degrees of the inputs and outputs in the entire \( (n, m) \) sparse crossbar, rather than its subgraphs.

Problem 2.47 [25] A printing spooler is to be built to serve 35 employee in a department. The department owns 10 dot-matrix printers, 10 300 dpi laser printers, 2 600 dpi printers, and 1 600 color printer that are interconnected by a sparse crossbar to the computers in employee offices. Employees are divided into three groups of 5, 10 and 20 people. Each employee in the first group is connected to a 600 dpi printer, a 300 dpi printer, and the color printer. Each employee in the 2nd group is connected to a 600 dpi printer, a 300 dpi printer, a dot matrix printer, and the color printer; and each employee in the third group is connected to a 300 dpi printer and a dot matrix printer. Due to technological constraints, at most 15 computers can be connected to the color printer, 8 to a 600 dpi printer, 5 to a 300 dpi or a dot-matrix printer.

(a) Assuming that each employee can issue a print command to one printer at a time, if all thirty employees issue a print command at once what is the minimum number of printers that can be matched with the employees under any given allowed choice of printers?

(b) If a fifth of the dot matrix printers and four-fifths of the 300 dpi laser printers are turned off for maintenance, what is the minimum number of employees that can print at once?
Problem 2.48  [25] A television video network is to supply pay per view movies from a collection of 100 movies to 100 stations across the United States that are divided into four groups of 25 stations. Within each viewing time slot, the stations in the first group can carry any one of 15 movies, the stations in the second group can carry any one of 20 movies, the stations in the third group can carry any one of 10 movies and the stations in the fourth group can carry any one of 40 movies. If the network can transmit each movie to only one of 20 stations within each viewing time slot, what is the minimum number of movies that can be carried on the hundred stations within the same viewing time slot?

Problem 2.49  [25] A set of 10 satellites are connected to 30 ground stations by microwave links. The ground stations are divided into three groups of 10 stations, $A$, $B$, and $C$. If each station in groups $A$ and $B$ is linked to 4 satellites, each station in group $C$ is linked to 2 satellites, and if no satellite is linked to more than 5 stations determine the largest number $x$ for which any $x$ stations – at least one of which is always in group $C$– can be linked to $x$ distinct satellites.

Problem 2.50  [35,Liu] Prove each of the following statements:

(a) Let $X_1$ and $X_2$ be any two subsets of inputs in a sparse crossbar. Then $\vartheta(X_1 \cup X_2) + \vartheta(X_1 \cap X_2) \geq \vartheta(X_1) + \vartheta(X_2)$.

(b) Let $X_1$ and $X_2$ be any two subsets of inputs in a sparse crossbar $G$ such that $\vartheta(X_1) = \vartheta(X_2) = \vartheta(G)$. Then $\vartheta(X_1 \cap X_2) = \vartheta(G)$.

(c) Use these two facts to prove Theorem 2.29.

Problem 2.51  [20]

(a) Determine the group of automorphisms of the binomial $G_{\binom{4}{2}}$-network.

(b) How many isomorphic copies does this network have?

Problem 2.52  [30]

(a) Determine the group of automorphisms of the fat-and-slim crossbar with $n$ inputs and $m$ outputs.

(b) Show that there exist $\binom{n}{m} m!$ isomorphic copies of the fat-and-slim crossbar.
Problem 2.53  [50]  
(a) Determine the group of automorphisms of the binomial $^m_2$-network.

(b) Determine the group of automorphisms of the binomial $^m_2$-network.

Problem 2.54  [50] Obtain an upper bound on the number of isomorphic copies of the binomial $^m_m$-network which is within a constant factor of the lower bound given in Theorem 2.31.

Problem 2.55  [35] Extend the upper bound in Theorem 2.31 to $^v^{v+2}$-networks.

Problem 2.56  [30]  
(a) Prove that if $G_1 = (A_1, B_1, E_1)$ and $G_2 = (A_2, B_2, E_2)$ are isomorphic sparse crossbars then, for each input $x \in A_1$ with out-degree $f_1$, there exists an input $x_2 \in A_2$ with out-degree $f_1$, and for each output $y \in B_1$ with in-degree $f_2$, there exists an output $y_2 \in B_2$ with in-degree $f_1$.

(b) Prove that the converse of the statement in part (a) is false by giving a counter example.

Problem 2.57  [40] Construct two sparse crossbar graphs $G_1$ and $G_2$ such that $\Xi_{G_1,m} \geq \Xi_{G_2,m}$ and $C_{G_1} < C_{G_2}$.

Problem 2.58  [40, Schrijver [BR91]] Let $A = [a_{ij}]$ be an $m \times n$ nonsingular binary matrix, where $m \leq n$, and let $r_1, r_2, \ldots, r_m$ denote the sums of columns in row 1, 2, $\ldots$, $m$, in that order. Prove that

$$\text{Per}(A) \leq \prod_{i=1}^{m} (r_i!)^{\frac{1}{r_i}}.$$ 

Problem 2.59  [30]  
(a) Let $N_{\text{unm}}$ denote the number of unordered $m$-matchings in an $n$-input and $m$-output sparse crossbar $G$. Show that

$$\text{Per}(A_G)/m! \leq N_{\text{unm}} \leq \text{Per}(A_G),$$
(b) Suppose that, for any fixed $m \times m$-submatrix $B$ of $A$, $\alpha \leq \text{Per}(B) \leq \beta$, where $\alpha$ and $\beta$ are nonnegative integers. Show that

$$\text{Per}(A) / \beta \leq N_{unm} \leq \text{Per}(A) / \alpha.$$ 

Problem 2.60 [30, Hall [BR91]] Let $A$ be an $m \times n$ binary matrix each row of which contains at least $r$ “1”s, where $r \geq m$. Show that $\text{Per}(A) \geq r!/(r-m)!$. Use this result and the solution of the previous problem to derive lower and upper bounds on the number of unordered matchings in an $n$-input and $m$-output fat-and-slim crossbar. How tight are your bounds?

Problem 2.61 [50] Obtain a closed form formula for the number of unordered $m$-matchings in an $n$-input and $m$-output sparse crossbar. What would be the implication of having such a formula?

Problem 2.62 [40, Hall [BR91]]

(a) Let $n_1, n_2, \ldots, n_m$ be integers with $n_1 \leq n_2 \leq \ldots \leq b_m$, a Ferrers matrix $F(n_1, n_2, \ldots, n_m)$ is an $n_m \times m$ binary matrix $F = [f_{i,j}]$, where $f_{i,j} = 1$ if and only if $1 \leq j \leq n_i, i = 1, 2, \ldots, m$. Prove that

$$\text{Per}(F(n_1, n_2, \ldots, n_m)) = \prod_{i=1}^{m} (n_i - i + 1).$$

(b) Show that if $n_1 = 1, n_2 = 2, \ldots, n_m = m$ then the Ferrer matrix $F(n_1, n_2, \ldots, n_m)$ is a lower triangular binary matrix, and use the formula in (a) to show that $F(n_1, n_2, \ldots, n_m)$ is an $(m, m)$-concentrator.

(Hint: If $F(n_1, n_2, \ldots, n_m)$ is a lower triangular $m \times m$ matrix then its permanent counts the number of unordered $m$-matchings in the corresponding sparse crossbar.)

Problem 2.63

(a) [20] Show that for any $m \times n$ matrix $A$,

$$\text{Per}(A) = \sum_{j=1}^{n} a_{i,j} \text{Per}(A_{i,j}),$$
where $A_{i,j}$ runs over all $(m - 1) \times (n - 1)$ minors of $A$ obtained by deleting its $i$th row and $j$th column.

(b) [50] Use the formula given in part (a) to compute the $\text{Per}(A_G)$, where $G$ is a fat-and-slim crossbar with $n$ inputs and $m$ outputs.

(c) [50] Repeat part (b) for the binomial $\binom{m}{m-2}$-network.

**Problem 2.64** [40] Determine the group of automorphisms of the $m$ input, $n$ output fat-and-slim crossbar with dilation $p$.

**Problem 2.65** [50] Obtain a closed form formula for the number of non-isomorphic sparse crossbar $(n,m,c)$-concentrators, $1 \leq c \leq m$.

**Problem 2.66** [40] Show that any $(n, m, m/2+1)$-concentrator requires at least $2n-m$ crosspoints. Can you construct an $(n, m, m/2+1)$-concentrator with $2n - m$ crosspoints?

**Problem 2.67** [50] Determine whether or not there exists a sparse crossbar $(n, m, c)$-concentrator whose crosspoint complexity matches the lower bound given in Theorem 2.15 within a constant factor for any $n \geq m \geq c$. If it does, construct one.

**Problem 2.68** [50]

(a) Compute the cardinality of $\text{XNC}(n,m)$, that is the number of $(n,m)$-sparse crossbars which are not an $(n,m)$-concentrator.

(b) Compute the cardinality of $\text{XC}(n,m)$, that is the number of sparse crossbar $(n,m)$-concentrators.

**Problem 2.69** [50]

Let $\text{SPC}(n, m, x)$ denote the set of sparse crossbar $(n,m)$ concentrators with each of its outputs having a fanin $x$. Clearly, $|\text{SPC}(n,m,x)| = 0$, for any $x \leq n - m$. Compute $|\text{SPC}(n,m,x)|$ for $x \geq n - m + 1$. 
