Equation of Continuity

The equation of continuity, \( \nabla \cdot \mathbf{J} + \frac{\partial \rho}{\partial t} = 0 \), is most intuitively understood in terms of the flow

of a compressible fluid such as air. Let the density of the air be \( \rho(x,y,z) \) and the velocity be given by \( \mathbf{V}(x,y,z) \) (boldface indicates a vector quantity). At any given point the flux of air, measured in grams per cm\(^2\) per sec is given by \( \mathbf{J}(x,y,z) = \rho(x,y,z) \mathbf{V}(x,y,z) \). Consider a small parallelepiped as shown. A small element of area \( d\mathbf{A} \) can be considered as a vector quantity by defining it as \( d\mathbf{A} = \mathbf{n} \, dA \) where \( \mathbf{n} \) is a unit vector whose direction is perpendicular to \( d\mathbf{A} \). For example, the element of area \( dA = dx \, dz \) on the right hand side of the parallelepiped becomes \( d\mathbf{A} = j \, dx \, dz \), where \( j \) is the unit vector along the y-axis. Suppose we take the element of area to be the one just defined (i.e., \( d\mathbf{A} \) is in the x-z plane). The flow of air mass through the area \( d\mathbf{A} \) measured in grams per sec is given by \( \mathbf{J} \cdot d\mathbf{A} = \rho(x,y,z) \mathbf{V} \cdot d\mathbf{A} = \rho(x,y,z) \mathbf{V}_y \, dx \, dz \) (grams per sec).

Now let \( d\mathbf{A} \) represent the surface area of the infinitesimal parallelepiped above. We calculate the net flow of mass out of the volume bounded by \( d\mathbf{A} \) (we take the normal vectors \( \mathbf{n} \) to be pointing out of the volume). At the left side the normal to the area \( dx \, dz \) points in the -y direction so \( d\mathbf{A} = -j \, dx \, dz \) and the mass flow is \(-J_y(x,y,z) dx \, dz\), where \( J_y(x,y,z) \) is the component of the flux in the y-direction at \( (x,y,z) \). The flow through the right side is \(+J(x,y+dy,z) dx \, dz\), where we now measure the flux at point \( (x, y+dy, z) \). To first order in the infinitesimals we find
\[ J_y(x, y+dy, z) = J_y(x, y, z) + \frac{\partial J_y}{\partial y} \ dy \] and the flow of matter (grams per sec) \textit{out} through the two sides perpendicular to the y-axis is given by

\[
\left( J_y(x, y, z) + \frac{\partial J_y(x, y, z)}{\partial y} \ dy - J_y(x, y, z) \right) \ dx \ dz = \frac{\partial J_y(x, y, z)}{\partial y} \ dx \ dy \ dz \ \text{(grams per sec)}.
\]

If this same calculation is done for the other faces of the parallelepiped we find the total flow of matter out of the little volume is

\[
\left( \frac{\partial I_x}{\partial x} + \frac{\partial I_y}{\partial y} + \frac{\partial I_z}{\partial z} \right) \ dx \ dy \ dz = \nabla \cdot \mathbf{J} \ dx \ dy \ dz \ (\text{grams per sec}).
\]

If we call this quantity \( M \) and define \( \,d\tau \) as the element of volume \( \,d\tau = dx \,dy \,dz \), then the change in mass per unit volume per second, or the change in density with time, is given by

\[
\frac{\partial M}{\partial \tau} = \nabla \cdot \mathbf{J} \ (\text{grams per cm}^3 \ \text{per sec}).
\]

Since the flow is out of the little volume, the change in density must be negative, which is just \(-\frac{\partial \rho}{\partial t}\). Hence we find \( \nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t} \) or

\[
\nabla \cdot \mathbf{J} + \frac{\partial \rho}{\partial t} = 0.
\]

In the case of the flow of electric charge we are dealing with a “fluid” of electrons or holes. The equation is the same as for a fluid of atoms.