Problem 1. Travelers buying airline tickets on a particular flight purchase economy class (40%), economy plus (35%) or business (25%). Of those in economy, 30% purchase duty-free goods in the onboard store, of those in economy plus 60% purchase duty-free, and of those in business 50%. Given that the next passenger bought duty-free, what is the probability that he is traveling in economy?

Solution: (Bayes formula)

\[ P(E_jD) = \frac{P(D_jE)P(E)}{P(D)} = \frac{P(D|E)P(E)}{P(D|E)P(E) + P(D|E+)P(E+) + P(D|B)P(B)} \]

\[ = \frac{0.3 \cdot 0.4}{0.3 \cdot 0.4 + 0.6 \cdot 0.35 + 0.5 \cdot 0.25} = \frac{24}{91} \]

Problem 2. A coin with \( Pr(H) = p \) is tossed repeatedly and independently until the first \( H \) is observed. Compute the probability of the event \( E \) that the first head appears in an even-numbered toss.

Solution 1: Let \( A_1 \) be the event that \( H \) appears in the first toss. We have

\[ P(E) = P(E|A_1)P(A_1) + P(E|A_c^1)P(A_c^1). \]

Of course, \( P(E|A_1) = 0 \), and \( E|A_c^1 \) is the event that \( H \) appears in an odd toss if we start counting from toss 2. We obtain \( P(E|A_c^1) = P(E^c) = 1 - P(E) \), and then from (1)

\[ P(E) = 0 \cdot p + (1 - P(E)) \cdot (1 - p) \]

which gives \( P(E) = (1 - p)/(2 - p) \).

Solution 2: The probability that \( H \) appears in toss \( 2k, k \geq 1 \) is \( (1 - p)^{2k-1}p \), so

\[ P(E) = \sum_{k=1}^{\infty} (1 - p)^{2k-1}p = \frac{p}{1 - p} \sum_{k=1}^{\infty} ((1 - p)^2)^k = \frac{p}{1 - p} \cdot \frac{(1 - p)^2}{1 - (1 - p)^2} \]

\[ = \frac{1 - p}{2 - p}. \]

Problem 3. Toss a fair coin 4 times and consider the random variable \( X \) indicating number of heads. Calculate \( P[X = x | X \text{ even}] \) for \( x = 0; 1; 2; 3; 4 \).

Solution: Let \( A = \{X \text{ even}\} \). We have

\[ P(A) = \frac{1}{16} \left( \binom{4}{0} + \binom{4}{2} + \binom{4}{4} \right) = \frac{1}{16}(1 + 6 + 1) = 1/2. \]
Then

\[ P[X = x | X \text{ even}] = \frac{P[X = x \text{ and } X \text{ even}]}{P[X \text{ even}]} = \begin{cases} 0, & x = 1, 3 \\ 2P[X = x], & x = 0, 2, 4 \end{cases} = \begin{cases} 0, & x = 1, 3 \\ \frac{1}{2}, & x = 0, 4 \end{cases} \frac{3}{4}, & x = 2 \] .

**Problem 4.** In an intersection, a car speeds to make the light with probability 0.2 and the decision to “go for it” is taken by different drivers independently. You have observed 6 cars passing through the intersection. Let \( X \) be the RV that equals the number of cars that sped to make the green light. (a) What is \( EX, \text{Var}(X) \)? (b) Find the probability that \( P(|X - EX| > 1) \) (approximate answer to within 10% is acceptable).

Solution: (a) \( X \) binomial with \( n = 6, p = 0.2 \), so \( EX = 1.2, \text{Var}(X) = 0.96 \).

(b)

\[
P(|X - EX| > 1) = 1 - P(|X - EX| \leq 1) = 1 - P(X \in \{1, 2\}) = 1 - 6 \cdot 0.2 \cdot 0.8^5 - \binom{6}{2} (0.2)^2 (0.8)^4
\]

\[
= 1 - 1.2 \cdot 0.32768 - 15 \cdot 0.04 \cdot 0.4096 \approx 1 - 0.38 - 0.25 = 0.37
\]

(the exact value is 0.361024).

**Problem 5.** I have 10 different apps on my phone. Someone hacked into it, and was able to make an app start every 6 minutes, but not to control which app starts, so every 6 minutes a random one out of the 10 launches itself. Assume that, unless I start using the app just launched, it’s immediately terminated, so they appear only for a very brief period of time.

(a) I decided to make use of this situation, and whenever I need an app I just wait for it to be launched. Suppose that I need app \( \#1 \). Let \( X \) be the number of apps that will be started before \( \#1 \) appears. What is the PMF of \( X \), and what is its expected value and variance? Note that \( EX \) and \( \text{Var}(X) \) should be expressed as numbers.

(b) I took the phone out of my pocket at a random, uniformly distributed time during the day. What’s the expected wait till the appearance of app \( \#1 \)?

Solution: (a) \( X \) is (almost) geometrically distributed: \( p_X(i) = \left( \frac{9}{10} \right)^i \frac{1}{10}, i = 0, 1, \ldots \). The “almost” comes from the fact that geometric is number of trials till first success including the trial that constitutes the success, while \( X \) is one less than that number. Denoting this geometric by \( Y \), we note that \( X = Y - 1 \), so

\[
EX = EY - 1 = \frac{1}{p} - 1 = 9; \quad \text{Var}(X) = \text{Var}(Y) = \frac{1 - p}{p^2} = \frac{9/10}{1/100} = 90.
\]

(b) It will take \( EU = 3 \) minutes on average for the first app to appear, where \( U \) is an RV uniformly distributed on the segment \([0, 6]\). After that, according to part (a), app \( \#1 \) will be the 10th on average, so the expected wait is \( 3 + 6 \cdot 9 = 57 \) min.