Max total 55 pts.  

- Please be precise and rigorous in your statements, and show all your calculations. Please write neatly and legibly. Be sure to Print Your Name!

- This is a closed book exam, but you are allowed up to two 8.5\(\times\)11 pages of notes. No calculators please! Good luck!

**Problem 1** (10pts)

Let the joint PMF of discrete RVs be given by

\[ p_{XY}(x,y) = \begin{cases} \frac{1}{25}(x^2 + y^2) & \text{if } x = 1, 2, \ y = 0, 1, 2 \\ 0 & \text{otherwise.} \end{cases} \]

Find \( p_{X|Y}(x|y) \), \( P(X = 2|Y = 1) \) and \( E(X|\{Y = 1\}) \).

**Solution:**

\[ p_Y(y) = p_{XY}(1,y) + p_{XY}(2,y) = \frac{1}{25}(2y^2 + 5), \quad y = 0, 1, 2. \]

\[ p_{X|Y}(x|y) = \frac{p_{XY}(x,y)}{p_Y(y)} = \frac{x^2 + y^2}{2y^2 + 5}, \quad y = 0, 1; x = 1, 2. \]

\[ P(X = 2|Y = 1) = p_{X|Y}(2|1) = 5/7. \]

\[ E(X|\{Y = 1\}) = 1 \cdot p_{X|Y}(1|1) + 2 \cdot p_{X|Y}(2|1) = \frac{2}{7} + 2 \cdot \frac{5}{7} = \frac{12}{7}. \]

**Problem 2** (10 pts)

Let \( X \) be an RV with PDF

\[ f_X(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{o/w} \end{cases} \]

for some \( \lambda > 0 \). Find the PDF of the RV \( Y = X^{2/3} \).

**Solution:** We have

\[ F_Y(y) = P(Y \leq y) = P(X^{2/3} \leq y) = P(X \leq y^{3/2}) = \int_0^{y^{3/2}} f_X(x) dx = 1 - e^{-\lambda y^{3/2}}, y \geq 0. \]

\[ f_Y(y) = F_Y'(y) = \frac{3\lambda}{2} \sqrt{y} e^{-\lambda y^{3/2}}, \quad y \geq 0. \]

**Problem 3** (10pts)

The joint pdf of RVs \( X \) and \( Y \) is given by

\[ f_{XY}(x,y) = \begin{cases} \frac{1}{16\pi} & \text{if } x^2 + y^2 \leq 16 \\ 0 & \text{o/w} \end{cases} \]

(a) Find \( f_X(x) \), \( E[X|Y] \)

(b) Let \( R = \sqrt{X^2 + Y^2} \). Find \( f_R(r) \) (hint: the pair \( X, Y \) is jointly uniformly distributed in the circle).

(c) Are \( X \) and \( Y \) independent?
**Solution:** (a) \( f_X(x) = \int_{-\sqrt{\frac{16-x^2}{8\pi}}}^{\sqrt{\frac{16-x^2}{8\pi}}} f_{XY}(x,t)dt = \frac{\sqrt{16-x^2}}{8\pi}, -4 \leq x \leq 4. \) By symmetry we have \( f_Y(y) = \frac{\sqrt{16-y^2}}{8\pi}, -4 \leq y \leq 4 \) and \( f_{X|Y}(x|y) = \frac{f_{XY}(x,y)}{f_Y(y)} = \frac{1}{2\sqrt{16-y^2}}, -\sqrt{16-y^2} \leq x \leq \sqrt{16-y^2}. \) Then for any \( y, -4 \leq y \leq 4 \)

\[
E[X|Y = y] = \int_{-\sqrt{16-y^2}}^{\sqrt{16-y^2}} \frac{xdx}{2\sqrt{16-y^2}} = 0
\]

and \( E[X|Y] = 0 \) with probability one (this is of course clear by symmetry without any computations).

(b) \[
F_R(r) = \begin{cases} 0 & \text{if } r \leq 0 \\ P(R \leq r) = P(X^2 + Y^2 \leq r^2) = \frac{\pi r^2}{16\pi} = \left(\frac{r}{4}\right)^2 & \text{if } 0 \leq r \leq 4 , \\ 1 & \text{if } r \geq 4 
\end{cases}
\]

and \( f_R(r) = F'_R(r) = r/8 \) if \( 0 \leq r \leq 4 \) and = 0 otherwise.

(c) From the definition and the PDFs computed above it is immediate that \( X \) and \( Y \) are not independent.

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**Problem 4 (15pts)**

Let \( X \sim \text{Unif}[0,1], Y \sim \text{exp}(\lambda) \) be independent RVs. Let \( Z = X + Y \).

(a) (10pts) Find the PDF \( f_Z(z) \) and the CDF \( F_Z(z) \).

(b) (5pts) Find \( E(Z^2) \) (Hint: This question can be solved independently of the solution of part (a)).

**Solution:** (a)

\[
f_Z(z) = \int_{-\infty}^{\infty} f_X(z-y)f_Y(y)dy.
\]

If \( 0 \leq z \leq 1 \), then \( f_X(z-y) \neq 0 \) for \( 0 \leq y \leq z \). If \( z \geq 1 \) then \( f_X(z-y) \neq 0 \) for \( z-1 \leq y \leq z \). We obtain

\[
f_Z(z) = \begin{cases} f_0^z \lambda e^{-\lambda y}dy = 1 - e^{-\lambda z}, & 0 \leq z \leq 1 \\ f_{z-1}^z \lambda e^{-\lambda y}dy = e^{-\lambda(z-1)} - e^{-\lambda z}, & z \geq 1. 
\end{cases}
\]

\[
F_Z(z) = \begin{cases} 0 & 0 \leq z \leq 1 \\ \int_0^z f_Z(t)dt = F_Z(1) + \int_1^z \left(e^{-\lambda(t-1)} - e^{-\lambda t}\right)dt \\ = F_Z(1) + \left(-\frac{1}{\lambda}\right)\left(e^{-\lambda(t-1)} - e^{-\lambda t}\right)|_1^z \\ = 1 + \frac{1}{\lambda}(e^{-\lambda} - 1) - \frac{1}{\lambda}(e^{-\lambda(z-1)} - e^{-\lambda z}) + \frac{1}{\lambda}(1 - e^{-\lambda}) \\ = 1 - \frac{1}{\lambda}(e^{-\lambda(z-1)} - e^{-\lambda z}), & z \geq 1.
\end{cases}
\]

(b) \( EX^2 = 1/3, EY^2 = \int_0^\infty y^2 e^{-\lambda y}/\lambda dy = -y^2 e^{-\lambda y}|_0^\infty + 2 \int_0^\infty ye^{-\lambda y}dy = 2/\lambda^2 \) (the last step because \( EY = 1/\lambda \)). Then

\[
E[X^2 + 2XY + Y^2] = \frac{1}{3} + \frac{2}{\lambda^2} + 2(EX)(EY)\text{(by independence)} = \frac{1}{3} + \frac{2}{\lambda^2} + \frac{1}{\lambda}.
\]

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**Problem 5 (10pts)**

Let \( X \) be an RV with pdf \( f_X(x) = xe^{-x} \) where \( x > 0 \). Calculate:

(1) The transform \( M_X(s) \) for \( X \).

(2) The moment \( E[X^n] \) using \( M_X(s) \) above.

Hint: the following relation could be useful

\[
\sum_{k=0}^{\infty} (k+1)x^k = \frac{1}{(1-x)^2}.
\]
Solution:

(1) \( M_X(s) = \int_0^\infty x e^{-x} e^{sx} dx = \int_0^\infty x e^{-(1-s)x} dx = \frac{1}{(1-s)^2}, \; s < 1 \)

(2) \( M_X(s) = (1-s)^{-2} = \sum_{k=0}^{\infty} (k+1)s^k = \sum_{k=0}^{\infty} \frac{(k+1)!}{k!} s^k \)

(Alternatively, \( M_X(s) = (1-s)^{-2} = \sum_{k=0}^{\infty} \left(\frac{-2}{k}\right)(-s)^k = \sum_{k=0}^{\infty} \frac{(k+1)!}{k!} s^k, \; s < 1 \) We have,

\[
\sum_{k=0}^{\infty} M_X^{(k)}(0) \frac{s^k}{k!} = \sum_{k=0}^{\infty} (k+1)s^k.
\]

Taking \( k = n \), we obtain \( \frac{M_X^{(n)}(0)}{n!} = n + 1 \) and

\[
E[X^n] = M_X^{(n)}(0) = (n + 1)!.
\]