# STRUCTURE AND DENSITY OF SPARSE CROSSBAR CONCENTRATORS ${ }^{1}$ 

By<br>Emre Gündüzhan and A. Yavuz Oruç<br>Electrical Engineering Department<br>University of Maryland<br>College Park, MD 20742


#### Abstract

:

A sparse crossbar ( $n, m$ )-concentrator is a bipartite graph with $n$ source and $m$ sink vertices, $m \leq n$, in which there exists a matching between every $m$ source vertices and the $m$ sink vertices. In this paper, we investigate the structure, and the density of sparse crossbar ( $n, m$ )-concentrators among all $2^{n m}$ bipartite graphs. We establish that the density of sparse crossbar concentrators is bounded from below by 0.2887 when $m=n$, from above by $1 / e$ when $m=n / 2$, and it tends to 0 when $m=1$, as $n \rightarrow \infty$. We also derive upper and lower bounds on the density of sparse crossbar $(n, m)$-concentrators for an arbitrary $m \leq n$. The lower bounds provide an insight into the structure of sparse crossbar concentrators, while the upper bounds give a partial characterization of bipartite graphs which fail to have a concentrator property.


[^0]
## 1 Introduction

Consider a device with $n$ distinguished terminals, called inputs, and $m$ distinguished terminals, called outputs, where each input is connected to some of the outputs by switches, called crosspoints. Such a device is called a sparse crossbar ( $n, m, c$ )concentrator if there exists a set of crosspoints by which every $c \leq m$ inputs can be connected to some $c$ outputs. The parameter $c$ is called the capacity of the concentrator. When $c=m$, the concentrator is said to have a full capacity, and we drop $c$ from the notation.

Sparse crossbar concentrators have been investigated in the literature in connection with efficient constructions of more powerful networks[Pip77,Bas81,JO93,MO94]. They have also been studied independently for their crosspoint complexity[Mas77, NM82] and structural properties[OH96,GO96]. It has been established that an ( $n, m, c$ )-concen- trator requires at least $(n-c+1) m /(m-c+1)$ crosspoints, and this bound is tight when $c=m[\mathrm{OH} 96]$. More specifically, it was shown that a sparse crossbar ( $n, m$ )-concentrator can be contructed with $(n-m+1) m$ crosspoints using a bipartite graph, called a fat-and-slim crossbar for any integers $n$ and $m \leq$ $n$. More recently, it was shown that sparse crossbar ( $n, m$ )-concentrators can be constructed with a minimum crosspoint complexity, and using nearly the same number of crosspoints per each input for arbitrary values of $n$ and $m \leq n[G O 96]$.

In this paper, we consider another interesting problem about sparse crossbar concentrators. The problem is to determine how little or abundantly the sparse crossbar full capacity concentrators are contained among all $2^{n m}$ sparse crossbars with $n$ inputs and $m$ outputs. This is a question of density, and as such, it is a measure of resilience of sparse crossbar concentrators among all bipartite graphs. In particular, the density of sparse crossbar $(n, m)$-concentrators among all $2^{n m}$ sparse crossbars represents the probability that a randomly constructed sparse crossbar is a concentrator. This density can also be viewed as the probability that a randomly constructed sparse crossbar concentrator remains a concentrator when some of its crosspoints fail to work. Aside from these practical motivations, the density problem is also interesting in its own right, and as we will see, it leads to a more complete understanding of the structure of sparse crossbar concentrators. In particular, the lower bounds on the density of sparse crossbar concentrators shed light on how such graphs are structured, while the upper bounds expose the reasons why
many bipartite graphs fail to be a concentrator.
The rest of the paper is organized as follows. In the next section, we present basic mathematical facts that will be needed in establishing our results. In Section 3, we summarize some of the recently constructed sparse crossbar concentrators. In Section 4, we formalize the density concept, and examine the density of sparse crossbar concentrators for a few small values of $n$ and $m$. In Section 5, we present upper and lower bounds on the density of sparse crossbar concentrators. The paper is concluded in Section 6 with the analysis and comparisons of the lower and upper bounds, and suggestions for future research.

## 2 Definitions and Preliminary Facts

We begin with some definitions.
Definition 1 : A sparse crossbar an ( $n, m$ )-concentrator is a bipartite graph with $n$ inputs and $m$ outputs such that there exists a matching between every $m$ of the inputs and the $m$ outputs.

Definition 2 : The incidence matrix of an ( $n, m$ )-concentrator is an $n \times m$ binary matrix $B=\left[b_{i, j}\right]_{m \times n}$ such that $b_{i, j}=1$ if and only if there exists an edge between input $j$ and output $i$.

Figure 1 illustrates the three representations of a sparse crossbar concentrator.
A key result which is central to the characterization of sparse concentrators is the bipartite graph version of Hall's theorem:

Theorem 1 : Let $G$ be a bipartite graph with $n$ inputs and $m \leq n$ outputs. $A$ subset of $m$ inputs has a matching in $G$ iff any $j$ inputs are connected to at least $j$ outputs, $1 \leq j \leq m$.

The next two results follow from Hall's theorem.

Theorem 2 : A bipartite graph with $n$ inputs and $m$ outputs is a sparse concentrator if and only if its incidence matrix does not contain $a(m-k+1) \times k$ zero submatrix, $k=1,2, \ldots, m$.


Fig. 1: Various representations of a sparse crossbar concentrator.
Proof: Let $G$ be a bipartite graph, and $B_{G}$ be its incidence matrix. If $B_{G}$ contains a $(m-k+1) \times k$ zero matrix then the $k$ inputs associated with this zero submatrix could at most be connected to

$$
m-(m-k+1)=k-1
$$

outputs, and any $m$ inputs that include those $k$ inputs cannot be matched with the $m$ outputs. On the other hand, suppose that $B_{G}$ does not contain any $(m-k+1) \times k$ zero submatrix. This implies that every $k$ inputs in $G$ are connected to at least $k$ outputs. Therefore, by Hall's theorem there must exist a matching between every $m$-inputs and the $m$ outputs. \|

Theorem 3 : In a sparse crossbar ( $n, m$ )-concentrator, every $k$ of the $m$ outputs must be connected ${ }^{2}$ to at least $n-m+k$ inputs.

Proof: If some $k$ outputs are connected only to $n-m+k-1$ or fewer inputs, then some $m-k+1$ or more inputs will be connected to only $m-k$ outputs, contradicting

[^1]that the graph in question is a concentrator. \|

## 3 Fat-and-slim and Banded Sparse Crossbars

One way to determine whether or not sparse crossbar concentrators occur abundantly among the set of all bipartite graphs is to construct families of sparse concentrators, and then compute their cardinalities. Two relatively large families of sparse crossbar concentrators have recently been introduced in the literature[OH96,GO96].

Definition 3 : Let $\mathcal{G}=(I, O)$ be a bipartite graph with $n$ inputs and $m$ outputs. Suppose that $I$ is partitioned into two sets $A_{1}$ and $A_{2}$, where $\left|A_{1}\right|=n-m$ and $\left|A_{2}\right|=m . \mathcal{G}$ is called an $(n, m)$-fat-and-slim crossbar if each of the $n-m$ inputs in $A_{1}$ is connected to all the $m$ outputs, and if each of the $m$ inputs in $A_{2}$ is connected to a single but distinct output.

A (12,4)-fat-and-slim crossbar is depicted in Figure 2.

Theorem 4 : Every ( $n, m$ )-fat-and-slim crossbar is an ( $n, m$ )-concentrator, and it uses the minimum number of crosspoints possible, (i.e. $(n-m+1) m$.)

Corollary 1 There exist at least

$$
\begin{equation*}
n \times(n-1) \times(n-2) \cdots \times(n-m+1) \tag{1}
\end{equation*}
$$

distinct sparse crossbar ( $n, m$ )-concentrators.
Proof: The $n-m$ slim section can be fixed in $\binom{n}{m}$ ways, and for each such slim section, the $m$ crosspoints can be distributed into the $m$ rows in $m$ ! ways. \|
By Stirling's approximation, the cardinality, $|F S(n, m)|$, of the family of fat-andslim crossbar concentrators is

$$
\begin{equation*}
|F S(n, m)| \approx\left(\frac{n-m}{e}\right)^{m}\left(\frac{n}{n-m}\right)^{n+0.5} e^{1 / 12 n-1 /(12 n+1)} \tag{2}
\end{equation*}
$$

When $m=n / 2$, this becomes

$$
\begin{equation*}
2^{n\left(1+\frac{\log _{2} e}{2}\right)+(n / 2) \log _{2}(n / 2)+0.5} \tag{3}
\end{equation*}
$$

and dividing this by $2^{n^{2}}$ shows that, while fat-and-slim ( $n, n / 2$ )-crossbars form a large family of sparse crossbar concentrators, their density tends to 0 as $n \rightarrow \infty$.


Fig. 2: A (12,4)-fat-and-slim crossbar.

Another family of sparse crossbar concentrators, called banded sparse crossbars can be obtained by placing blocks of $n-m+1$ " 1 "s into the $m$ rows of an $m \times n$ incidence matrix.

Definition 4 An ( $n, m$ )-sparse crossbar $G$ is called (column) banded if its incidence matrix $B_{G}=\left[b_{i, j}\right]$ is given by

$$
b_{i, j}=\left\{\begin{array}{l}
1 \quad \text { if } i \leq j \leq i+n-m \\
0 \quad \text { if } j<i \text { or } j>i+n-m
\end{array}\right.
$$

for $i=1,2, \ldots, m$.

Theorem 5 : Every banded ( $n, m$ ) sparse crossbar is a concentrator.

The theorem can be proved using the following lemma[GO96].

Lemma 1 (Guo-Oruç:) Let $X$ and $Y$ be any two columns of an $n \times m$ binary matrix $A$. Suppose $X$ covers $^{3} Y$, and let $a_{i_{1}, x}, \ldots, a_{i_{r}, x}$ be any $r$ rows of " 1 "s in $X$. Let $B$ be a matrix obtained from $A$ by exchanging $a_{i_{l}, x}$ with $a_{i_{l}, y}, 1 \leq l \leq r$. If $A$ does not have a $(m-k+1) \times k$ zero submatrix, for any $k, 1 \leq k \leq m$, then neither does $B$.

Definition 5 : Two sparse crossbar concentrators $G_{1}$ and $G_{2}$ are said to be topologically equivalent if $G_{1}$ can be obtained from $G_{2}$ by permuting the inputs and/or outputs of $G_{2}$, and conversely.

[^2]

Fig. 3: A banded sparse crossbar concentrator.
Definition 6 : Any sparse crossbar concentrator which is topologically equivalent to a banded sparse crossbar is also called banded.

Corollary 2 : For $2 \leq m \leq n / 2$, there exist at least

$$
\begin{equation*}
\binom{n}{n-2 m+2}(2 m-2)!=n \times(n-1) \times(n-2) \cdots \times(n-2 m+3) \tag{4}
\end{equation*}
$$

distinct banded sparse crossbar concentrators.

Proof: The $n-2 m+2$ full columns (i.e., those containing only " 1 " entries) of a banded sparse crossbar concentrator can be fixed in $\binom{n}{n-2 m+2}$ ways, and any permutation of the remaining $2 m-2$ columns gives a distinct banded sparse crossbar concentrator. ||

Comparing (4) with (1) shows that the set of banded sparse crossbars is larger than the set of fat-and-slim crossbars when $n-2 m+3<n-m+1$, or $m>2$. However, it can be shown that the density of banded sparse crossbar concentrators also tends to 0 as $n \rightarrow \infty$.

We can construct a larger family of sparse concentrators, by increasing the number of " 1 "s in each column to $n-2 m+2 i, 1 \leq i \leq m$. It is obvious that the resulting sparse crossbar for each $i, 1 \leq i \leq m$ is a concentrator, and as in the proof of the corollary, each permutation of the remaining $2 m-2 i$ columns will result in a distinct concentrator after the $n-2 m+2 i$ full columns are fixed. Hence we have the following theorem:

Theorem 6 For $2 \leq m \leq n / 2$, there exists a family of sparse crossbar concentrators with no less than

$$
\sum_{i=1}^{m}\binom{n}{n-2 m+2 i}(2 m-2 i)!
$$

elements.
It is easy to verify that this set of sparse crossbar concentrators does not overlap with the set of fat-and-slim sparse crossbars.

## 4 The Density of Sparse Crossbar Concentrators

We have described two families of sparse crossbar concentrators and the two together form a set of

$$
\left(\prod_{j=1}^{m} n-j+1\right)+\left(\sum_{i=1}^{m} \prod_{j=n-2 m+2 i+1}^{n} j\right)
$$

elements. The main problem we now wish to deal with is "what portion of all the $2^{n m}$ bipartite graphs are actually sparse crossbar ( $n, m$ )-concentrators?" Formally, we are interested in the density of sparse crossbar concentrators among the $2^{n m}$ sparse crossbars with $n$ inputs and $m$ outputs, where:

Definition 7 : The density $d_{C(n, m)}$ of the set of sparse crossbar ( $n, m$ )-concentrators, $C(n, m)$, among all sparse crossbars with $n$ inputs and $m$ outputs is given by:

$$
d_{C(n, m)}=\operatorname{Lim}_{n \rightarrow \infty}|C(n, m)| \times 2^{-n m}
$$

It is easy to see that when $m=1$, there is only one sparse crossbar concentrator for all $n \geq 1$, and hence

$$
d_{C(n, 1)}={ }_{n \rightarrow \infty}^{\operatorname{Lim}} 2^{-n}=0
$$

For other values of $m$, the problem is not as trivial. For example, suppose $m=n$. In this case, we are dealing with square matrices, and to count all $n \times$ mattrices in which there is a matching, we must have an actual algorithm to construct all such matrices. For $n=2$, it is easy to see that of the $162 \times 2$ binary matrices, exactly 7 correspond to sparse crossbar concentrators. For $n=3$, we can count them by examining the structure of $3 \times 3$ matrices that correspond to sparse crossbar ( 3,3 )concentrators. As a first step, consider those $3 \times 3$ matrices in which the first row contains a single " 1 ". For this matrix to correspond to a concentrator, it must have a " 1 " in its second row which is located in a different column than the one which
contains the 1 in the first row. Likewise, the third row must contain a " 1 " in a column other than the columns in which the first and second rows had their " 1 "s. With this observation, it is readily seen that, when the first row is 001 , the structure of the concentrator matrices must conform to one of the five patterns, where the "-" entries indicate that the entry can be 0 or 1 .

$$
\left[\begin{array}{ccc}
0 & 0 & 1 \\
0 & 1 & - \\
1 & - & -
\end{array}\right],\left[\begin{array}{ccc}
0 & 0 & 1 \\
1 & 0 & - \\
- & 1 & -
\end{array}\right],\left[\begin{array}{ccc}
0 & 0 & 1 \\
1 & 1 & - \\
0 & 1 & -
\end{array}\right],\left[\begin{array}{ccc}
0 & 0 & 1 \\
1 & 1 & - \\
1 & 0 & -
\end{array}\right],\left[\begin{array}{ccc}
0 & 0 & 1 \\
1 & 1 & - \\
1 & 1 & -
\end{array}\right] .
$$

Noting that the "-" entries can be replaced with 0 or 1, we find that there exist 28 $3 \times 3$ concentrators whose first row is 001 . Likewise, each of the $3 \times 3$ matrices whose first row contains 010 or 100 contribute 28 to the total count. The next group of $3 \times 3$ matrices which correspond to concentrators are those whose first row contains exactly two " 1 "s. Consider those matrices whose first row contains 011 . It can be verified that if the second row contains two " 1 " s , and the locations of these " 1 "s do not coincide with the columns in the first row then the third row can contain any pattern of " 0 "s and " 1 "s, except 000 . In case that both " 1 "s are in the same columns as those in the first row, then the third row must contain a ' 1 ' in the first column, and the other entries can be 0 or 1 . Analyzing the other cases similarly, we have the following patterns of $3 \times 3$ matrices when the first row is 011 , and where $x$ entries indicate that the third row can contain any pattern of " 0 "s and " 1 "s except all "0"s.

$$
\begin{gathered}
{\left[\begin{array}{ccc}
0 & 1 & 1 \\
0 & 0 & 1 \\
1 & - & -
\end{array}\right],\left[\begin{array}{lll}
0 & 1 & 1 \\
0 & 1 & 0 \\
1 & - & -
\end{array}\right],\left[\begin{array}{ccc}
0 & 1 & 1 \\
1 & 0 & 0 \\
1 & - & -
\end{array}\right],\left[\begin{array}{ccc}
0 & 1 & 1 \\
1 & 0 & 1 \\
- & 0 & 1
\end{array}\right],\left[\begin{array}{ccc}
0 & 1 & 1 \\
1 & 1 & 0 \\
- & 1 & 0
\end{array}\right],\left[\begin{array}{ccc}
0 & 1 & 1 \\
1 & 0 & 1 \\
- & 1 & 1
\end{array}\right],} \\
{\left[\begin{array}{lll}
0 & 1 & 1 \\
1 & 0 & 1 \\
x & x & x
\end{array}\right],\left[\begin{array}{lll}
0 & 1 & 1 \\
1 & 1 & 0 \\
x & x & x
\end{array}\right]\left[\begin{array}{lll}
0 & 1 & 1 \\
1 & 1 & 1 \\
x & x & x
\end{array}\right] .}
\end{gathered}
$$

The first three patterns contribute 4 concentrators each, the next three contributes 2 concentrators each, and the last three contributes 7 concentrators each, resulting in a total of 39 concentrators. Similarly, the other two $3 \times 3$ matrices whose first row contains 101 or 110 will each contribute 39 to the total count. Finally, the following patterns of matrices whose first row contains 111 must be taken into account, where again the $x$ entries can be any pattern of " 0 "s and " 1 "s except all " 0 " $s$, and the
"-" " entries can be 0 or 1 .

$$
\begin{gathered}
\\
\\
{\left[\begin{array}{ccc}
1 & 1 & 1 \\
0 & 0 & 1 \\
x & x & -
\end{array}\right],\left[\begin{array}{lll}
1 & 1 & 1 \\
0 & 1 & 0 \\
x & - & x
\end{array}\right],\left[\begin{array}{ccc}
1 & 1 & 1 \\
1 & 0 & 0 \\
- & x & x
\end{array}\right],} \\
{\left[\begin{array}{lll}
1 & 1 & 1 \\
0 & 1 & 1 \\
x & x & x
\end{array}\right],\left[\begin{array}{lll}
1 & 1 & 1 \\
1 & 0 & 1 \\
x & x & x
\end{array}\right],\left[\begin{array}{lll}
1 & 1 & 1 \\
1 & 1 & 0 \\
x & x & x
\end{array}\right]\left[\begin{array}{lll}
1 & 1 & 1 \\
1 & 1 & 1 \\
x & x & x
\end{array}\right] .}
\end{gathered}
$$

Adding these matrices up we find that they contribute $3 \times 6+4 \times 7=46$ concentrators to the total count. Hence, the total number of sparse crossbar $(3,3)$-concentrators is given by

$$
3 \times 28+3 \times 39+46=247
$$

By extending these counting arguments, it can be shown that the total number of sparse crossbar (4,4)-concentrators is given by 37811. These numbers suggest that $d_{C(n, n)} \rightarrow 1$ as $n \rightarrow \infty$, but the most we can show that $d_{C(n, n)} \geq 0.288$ as $n \rightarrow \infty$.

Theorem 7 The density of sparse crossbar ( $n, n$ )-concentrators is at least 0.288 as $n \rightarrow \infty$.

Proof: The proof will follow from a more generalized statement in the next section.

## 5 Density Bounds

In this section, we present two upper bounds and two lower bounds on the density of sparse crossbar concentrators.

## A. Upper Bounds:

The most obvious upper bound can be derived using Theorem 3 with $k=1$. The theorem states that every sparse crossbar $(n, m)$-concentrator must have at least $(n-m+1) m$ crosspoints. It follows that all $m \times n$ matrices with fewer than $(n-m+1) m$ " 1 "s should correspond to bipartite graphs which are not concentrators. Hence,

$$
\begin{equation*}
d_{C(n, m)} \leq \operatorname{Lim}_{n \rightarrow \infty} \frac{2^{n m}-\sum_{i=0}^{(n-m+1) m-1}\binom{n m}{i}}{2^{n m}} \tag{5}
\end{equation*}
$$

or

$$
\begin{equation*}
d_{C(n, m)} \leq \operatorname{Lim}_{n \rightarrow \infty} \frac{\sum_{i=(n-m+1) m}^{n m}\binom{n m}{i}}{2^{n m}} \tag{6}
\end{equation*}
$$

Using Moon's inequality[EJ71],

$$
\sum_{i>r / 2+\lambda}^{r}\binom{r}{i}<2^{r} e^{-2 \lambda^{2} / r},
$$

in the interval $0 \leq \lambda \leq r / 2$, with $r=n m, \lambda=n m / 2-m(m-1)$, yields the upper bound

$$
\begin{equation*}
d_{C(n, m)} \leq \operatorname{Lim}_{n \rightarrow \infty} e^{-2\left((n m / 2-m(m-1))^{2} / n m\right.} \tag{7}
\end{equation*}
$$

or

$$
\begin{equation*}
d_{C(n, m)} \leq \operatorname{Lim}_{n \rightarrow \infty} e^{-2 m\left((n / 2-(m-1))^{2} / n\right.} \tag{8}
\end{equation*}
$$

in the interval $1 \leq m \leq n / 2+1$. It follows that

## Corollary 3 :

$$
d_{C(n, n / 2)} \leq 1 / e \approx 0.37
$$

We can derive another upper bound on $d_{C(n, m)}$ in the same interval by lower bounding the number of bipartite graphs which are not concentrators as above. This time, we count those bipartite graphs in any row of which there are $m$ or more zeros. Again, by Theorem 3, these correspond to bipartite graphs which are not concentrators.

Let $R_{k}$ denote the set of $m \times n$ matrices with $m$ or more zeros in the $k$ th row. Let $\left|R_{k}\right|$ denote the number of elements in $R_{k}$. Then,

$$
\begin{aligned}
\left|R_{j_{1}}\right|= & \sum_{i=m}^{n}\binom{n}{i} 2^{n(m-1)}, \\
\left|R_{j_{1}} \cap R_{j_{2}}\right|= & \left(\sum_{i=m}^{n}\binom{n}{i}\right)^{2} 2^{n(m-2)}, \\
& \vdots \\
\left|R_{j_{1}} \cap \cdots \cap R_{j_{m}}\right|= & \left(\sum_{i=m}^{n}\binom{n}{i}\right)^{m},
\end{aligned}
$$

for any distinct indices $j_{1}, \ldots, j_{m}$.
We can use these equalities to derive a lower bound on the number, $x_{C(n, m)}$, of bipartite graphs which are not concentrators with the inclusion exclusion principle.

$$
x_{C(n, m)} \geq\left|R_{1} \cup R_{2} \cup \cdots \cup R_{m}\right|
$$

$$
\begin{aligned}
& =\sum_{i}\left|R_{i}\right|-\sum_{i, j}\left|R_{i} \cap R_{j}\right|+\cdots+(-1)^{m+1}\left|R_{1} \cap \cdots \cap R_{m}\right| \\
& =\sum_{j=1}^{m}\binom{m}{j}(-1)^{j+1}\left(\sum_{i=m}^{n}\binom{n}{i}\right)^{j} 2^{n(m-j)} \\
& =-2^{n m} \sum_{j=1}^{m}\binom{m}{j}\left(\sum_{i=m}^{n}\binom{n}{i}\left(-2^{-n}\right)\right)^{j} .
\end{aligned}
$$

Using the equality

$$
\sum_{j=1}^{m}\binom{m}{j} x^{j}=(x+1)^{m}-1
$$

we get

$$
\begin{aligned}
x_{C(n, m)} & \geq-2^{n m}\left(\left(\sum_{i=m}^{n}\binom{n}{i}\left(-2^{-n}\right)+1\right)^{m}-1\right) \\
& =2^{n m}-\left(2^{n}-\sum_{i=m}^{n}\binom{n}{i}\right)^{m} \\
& =2^{n m}-\left(\sum_{i=0}^{m-1}\binom{n}{i}\right)^{m}
\end{aligned}
$$

Since $d_{C(n, m)}=1-\operatorname{Lim}_{n \rightarrow \infty}\left(x_{C(n, m)} / 2^{n m}\right)$, we get

$$
\begin{equation*}
d_{C(n, m)} \leq \operatorname{Lim}_{n \rightarrow \infty}\left(\frac{1}{2^{n}} \sum_{i=0}^{m-1}\binom{n}{i}\right)^{m} \tag{9}
\end{equation*}
$$

Rewriting the inequality as

$$
\begin{equation*}
d_{C(n, m)} \leq \operatorname{Lim}_{n \rightarrow \infty}\left(\frac{1}{2^{n}} \sum_{i=n-m+1}^{n}\binom{n}{i}\right)^{m} \tag{10}
\end{equation*}
$$

and once again, using Moon's inequality with $\lambda=(n-m+1)-n / 2$, we obtain the same upper bound

$$
\begin{equation*}
d_{C(n, m)} \leq \operatorname{Lim}_{n \rightarrow \infty} e^{-2 m(n / 2-m+1)^{2} / n} \tag{11}
\end{equation*}
$$

in the interval $1 \leq m \leq n / 2+1$
As we will see in the next section, if the second bound is computed directly using the expression in Eqn. (10) it provides a tighter upper bound on $d_{C(n, m)}$. This bound can be improved by considering larger values of $k$ in Theorem 2. However, the counting gets more complicated, and lest we count these non-concentrator graphs due to the higher order terms precisely, their contribution is negligible on $x_{C(n, m)}$.

## B. Lower Bounds

To find a lower bound on $d_{C(n, m)}$, we count the number of $m \times n$ matrices that correspond to an $(n, m)$-concentrator, and can be generated from a fat-and-slim concentrator. We use the transformation lemma (Lemma 1) that has been stated before.

First assume that $m \leq n-m$. This makes the fat region of the concentrator at least as large as the slim region. Suppose we choose any $i$ columns from the fat region and any $i$ columns from the slim region, and swap some entries between the first columns selected from each group, then between second columns selected from each group, and so on. For each pair of columns, there can be at most $m-1$ swaps, yielding to $2^{m-1}-1$ different arrangements of the two columns (we exclude the case of no swaps). This gives $\left(2^{m-1}-1\right)^{i}$ different concentrators for each selection of the groups. Summing this up for all the possible selections, we get

$$
|C(n, m)| \geq \sum_{i=0}^{m}\binom{m}{i}\binom{n-m}{i}\left(2^{m-1}-1\right)^{i}
$$

Now, suppose we add $k$ columns to the fat region, and remove $k$ columns from the slim region. For each $k$, we will get a different concentrator since the total number of crosspoints are changed. We can use the above generation process on the concentrators obtained in this way, which modifies the bound as

$$
|C(n, m)| \geq \sum_{k=0}^{m} \sum_{i=0}^{m-k}\binom{m-k}{i}\binom{n-m+k}{i}\left(2^{m-1}-1\right)^{i} .
$$

We can use the same counting technique for the $m>n-m$ case. In this case, the summation limits will be different. Also, increasing the fat region will eventually bring the problem to the previous case, so there will be two summation terms. The general bound for $|C(n, m)|$ is given by

$$
|C(n, m)| \geq \begin{cases}\sum_{k=0}^{m} \sum_{i=0}^{m-k}\binom{m-k}{i}\binom{n-m+k}{i}\left(2^{m-1}-1\right)^{i}, & \mathrm{~m} \leq \mathrm{n}-\mathrm{m} \\ \sum_{k=0}^{m-n / 2} \sum_{i=0}^{n-m+k}\binom{m-k}{i}\binom{n-m+k}{i}\left(2^{m-1}-1\right)^{i} & \\ +\sum_{k=1}^{n / 2} \sum_{i=0}^{n / 2-k}\binom{n / 2-k}{i}\binom{n / 2+k}{i}\left(2^{m-1}-1\right)^{i}, & \mathrm{~m}>\mathrm{n}-\mathrm{m}\end{cases}
$$

and

$$
d_{C(n, m)} \geq \operatorname{Lim}_{n \rightarrow \infty} \frac{|C(n, m)|}{2^{n m}}
$$

This lower bound can be improved using another construction. The key idea is to ensure that each input has a distinct output to be matched with. This is guaranteed if we enter a " 1 " in a different column in each row as we prove in the next theorem.

Theorem 8 : There exist at least

$$
\begin{equation*}
\left(\prod_{i=1}^{m-1} 2^{i}\left(2^{m-i}-1\right)\right) \times\left(\sum_{i=n-m+1}^{n}\binom{n}{i}\right) \tag{12}
\end{equation*}
$$

sparse crossbar concentrators.
Proof: We know from Theorem 3 that every row in the incidence matrix of a sparse crossbar concentrator must have at least $n-m+1$ " 1 "s. Hence, the first row can be fixed in any one of

$$
\sum_{i=n-m+1}^{n}\binom{n}{i}
$$

ways. Once the first row is fixed this way, we fill some $n-m$ columns which have " 1 " entries in the first row with " 1 "s, and mark a " 1 " among the remaining " 1 "s in the first row. This insures that the input which corresponds to the column with the marked "1" can be matched with an output. This process can be iterated by allowing the remaining $m-1$ columns in the second row to assume any one of $2^{m-1}-1$ patterns of " 1 's and " 0 "s (we only need to exclude the all " 0 " pattern). Furthermore, the entry at the intersection of the second row and the column in which the 1 has been marked can be either 0 or 1 . Hence, the unspecified $m$ entries in the second row can be fixed in any one of $2\left(2^{m-1}-1\right)$ ways. Iterating this argument for the next row, and so on, we see that the total number of sparse crossbar ( $n, m$ )-concentrators is not less than the expression given in 12 . ||

Remark 1 When $n=m$, the lower bound in (12) reduces to

$$
\begin{equation*}
\prod_{i=0}^{n-1} 2^{i}\left(2^{n-i}-1\right)=2^{n^{2}} \prod_{i=1}^{n}\left(1-2^{-i}\right) \tag{13}
\end{equation*}
$$

Dividing this expression by $2^{n^{2}}$, we find that the density of a sparse crossbar ( $n, n$ )concentrator satisfies the inequality

$$
\begin{equation*}
d_{C(n, n)} \geq \prod_{i=1}^{n}\left(1-2^{-i}\right) \tag{14}
\end{equation*}
$$

Using the Euler expansion [BR42]

$$
\prod_{i=1}^{\infty}\left(1-q^{i}\right)=1-\left(q^{+} q^{2}\right)+\left(q^{5}+q^{7}\right)-\left(q^{12}+q^{15}\right)+\ldots
$$

with $q=1 / 2$, the indices being alternately $n(3 n \pm 1) / 2$, we find that

$$
d_{C(n, m)} \geq 1-(1 / 2+1 / 4)+(1 / 32+1 / 128)-(1 / 1024+1 / 32768) \geq 0.288
$$

## 6 Concluding Remarks

The exact values of the lower and upper bounds are plotted in Figures 4, 5 and 6 . It is seen that the second upper bound is always tighter than the first upper bound. On the other hand, the first lower bound is slightly tighter than the second lower bound when $m=0.2 n$, but the second lower bound gets tighter as $m$ approaches $n$. In fact, when $m=n$, it tends to 0.288 , whereas the first lower bound tends to 0 .

In this paper, we have investigated the density of sparse crossbar concentrators within the set of all bipartite graphs with $n$ inputs and $m$ outputs. The results we presented here offer some clues on how extensively such graphs are found among bipartite graphs. Perhaps, the most startling fact that has been uncovered is that the density of sparse crossbar concentrators is strictly greater than 0 (in fact, it is not less than 0.288 when the number of inputs equals the number of outputs.) The exact values of the density for $n=2,3,4$ show that it is likely that it tends to 1 as $n \rightarrow 1$, but proving this remains an open problem. For the more general case ( $n \neq m$,) the lower and upper bounds we have given in the paper seem to have room for improvement. For the lower bounds, it seems that the contructions given in the paper can be refined to more accurately count the number of concentrators. Improving the upper bounds by way of counting the bipartite graphs which are not concentrators appears to be more intractable.


Fig. 4: Lower and upper bounds when $m=0.2 n, 1 \leq n \leq 40$.


Fig. 5: Lower and upper bounds when $m=0.5 t n, 1 \leq n \leq 40$.


Fig. 6: Lower and upper bounds when $m=0.8 n, 1 \leq n \leq 40$.

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## OUTLINE

- A. Problems
- B. Mathematical Characterization
- C. Fat-and-Slim Crossbar Concentrators
- D. Banded Sparse Crossbar Concentrators
- E The Density Question
- F. Plots
- G. Concluding Remarks


[^0]:    ${ }^{1}$ This work is supported in part by the National Science Foundation under grant No. NCR9405539.

[^1]:    ${ }^{2}$ We say that $k$ outputs are connected to $r$ inputs if each of the $r$ inputs is connected to at least one of the $k$ outputs.

[^2]:    ${ }^{3}$ We say that column $X$ covers column $Y$ if whenever a row in $Y$ contains 1 so does $X$.

