Iterative Signature Adaptation for Capacity Maximization of CDMA Systems*

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Abstract

For single cell synchronous code division multiple access (CDMA) systems, both the information theoretic capacity and the network capacity have been identified. In both cases, it was shown that if the number of users, N, is no more than the processing gain, L, then orthogonal signatures are optimal while if $N > L$, then signatures which satisfy the Welch bound on the total squared correlation with equality (called WBE sequences) are optimal. This paper presents an algorithm which iteratively updates the signatures in a distributed fashion, starting from an initial set of signatures. Under mild conditions on the initial set of signatures, we prove that the algorithm converges to a set of orthogonal signatures if $N \leq L$ and to a WBE set if $N > L$. At each step, the algorithm replaces one signature from the set with the normalized linear MMSE receiver filter corresponding to that signature. Since the MMSE filter can be obtained by a distributed algorithm for each user, the algorithm is amenable to distributed implementation.

1 Introduction

In CDMA systems, users modulate their information streams with high frequency waveforms called CDMA codes or signature sequences. The signature waveform of user i, denoted by $s_i(t)$, is non-zero only in the bit interval $[0, T_b]$ and is normalized to unit energy, i.e., $\int_0^{T_b} s_i^2(t)dt = 1$. The received base band signal at the base station is

$$r(t) = \sum_{i=1}^{N} \sqrt{p_i} b_i s_i(t) + n(t) \tag{1}$$

where p_i and b_i are the received power and the information bit of user i, and $n(t)$ is the additive white Gaussian noise (AWGN) with zero mean and power spectral density σ^2.

We define the chip waveform to be $\psi(t)$, $t \in [0, T_c]$ and 0 elsewhere, where T_c is the chip duration. Thus $\{\psi(t - iT_c), i = 0, \ldots, L-1\}$, where $L = T_b/T_c$ is the processing gain, is a basis for the signal space. This allows us to represent the signature sequences of the users with L dimensional vectors. We will use $\boldsymbol{s}_i$ to denote the signature sequence of user i. We define an $L \times N$ matrix $\boldsymbol{S}$ which contains the signature sequences of all users in its columns, i.e., $\boldsymbol{S} = [\boldsymbol{s}_1, \boldsymbol{s}_2, \cdots, \boldsymbol{s}_N]$ and a $N \times N$ diagonal matrix $\boldsymbol{P}$ whose ith diagonal element is p_i. In terms of signal vectors, the received signal can be written as

$$\boldsymbol{r} = \sum_{i=1}^{N} \sqrt{p_i} b_i \boldsymbol{s}_i + \boldsymbol{n} \tag{2}$$

where $\boldsymbol{n}$ is a Gaussian random vector with $E\left[\boldsymbol{n}\boldsymbol{n}^\top\right] = \sigma^2 \boldsymbol{I}_L$, where $\boldsymbol{I}_L$ is the $L \times L$ identity matrix.

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The capacity of a single cell synchronous CDMA system has been studied in two different contexts. In [1,2], the information theoretic capacity region of a CDMA channel is

$$\mathcal{C} = \bigcap_{J \subseteq \{1,\cdots,N\}} \left\{ (R_1, \cdots, R_N) : 0 \leq \sum_{j \in J} R_j \leq \frac{1}{2} \log \left[\det \left(\boldsymbol{I}_L + \sigma^{-2} \boldsymbol{S}_J \boldsymbol{P}_J \boldsymbol{S}_J^\top \right) \right] \right\} \tag{3}$$

where $\boldsymbol{S}_J$ is the $L \times |J|$ matrix which is obtained from $\boldsymbol{S}$ by striking out the columns of $\boldsymbol{S}$ whose indices do not belong to the subset J, and $\boldsymbol{P}_J$ is a $|J| \times |J|$ diagonal matrix which is obtained from $\boldsymbol{P}$ by striking out the rows and columns of $\boldsymbol{P}$ whose indices do not belong to the subset J.

An important measure of overall information capacity of a multiaccess channel is the *sum* capacity

$$C_{\text{sum}} = \max_{(R_1,\cdots,R_N) \in \mathcal{C}} \sum_{i=1}^{N} R_i \tag{4}$$

From (3), the sum capacity of a CDMA channel is

$$C_{\text{sum}} = \frac{1}{2} \log \left[\det \left(\boldsymbol{I}_L + \sigma^{-2} \boldsymbol{S} \boldsymbol{P} \boldsymbol{S}^\top \right) \right] \tag{5}$$

When the powers of the users are the same, $p_i = p$ for all i, (5) reduces to

$$C_{\text{sum}} = \frac{1}{2} \log \left[\det \left(\boldsymbol{I}_L + \frac{p}{\sigma^2} \boldsymbol{S} \boldsymbol{S}^\top \right) \right] = \frac{1}{2} \log \left[\det \left(\boldsymbol{I}_N + \frac{p}{\sigma^2} \boldsymbol{S}^\top \boldsymbol{S} \right) \right] \tag{6}$$

where the last equality follows from the fact that for any two matrices $\boldsymbol{A}_{K \times M}$ and $\boldsymbol{B}_{M \times K}$,

$$\det \left(\boldsymbol{I}_K + \boldsymbol{A}\boldsymbol{B} \right) = \det \left(\boldsymbol{I}_M + \boldsymbol{B}\boldsymbol{A} \right) \tag{7}$$

Clearly, the sum capacity depends on the signature sequence set being used. It was shown in [2] that the sum capacity for equal received powers is maximized if the signature sequences are chosen such that if $N \leq L$,

$$\boldsymbol{S}^\top \boldsymbol{S} = \boldsymbol{I}_N \tag{8}$$

and if $N > L$,

$$\boldsymbol{S} \boldsymbol{S}^\top = \frac{N}{L} \boldsymbol{I}_L \tag{9}$$

The signature sequence sets satisfying (8) contain N orthogonal signature sequences in L dimensional vector space. When $N \leq L$, N such sequences can be found. The sequence sets satisfying (9) are named Welch Bound Equality (WBE) sequence sets in [2] because they satisfy the Welch's bound on the sum of the squares of the cross correlations of unit energy sequences with equality.

In [3], the *network* capacity of a CDMA system is defined in terms of the maximum number of admissible users. Given the processing gain L and a common Signal to Interference Ratio (SIR) target β, N users are said to be admissible if there exist positive powers p_i and signature sequences $\boldsymbol{s}_i$ such that each user has an SIR at least as large as the target SIR β. The network capacity was found for two kinds of linear receiver structures in [3]: matched filters and Minimum Mean Squared Error (MMSE) filters [4,5]. The MMSE filter for the kth user, $\boldsymbol{c}_k$, minimizes the MSE between the bit and the filter output of user k

$$\mathrm{MSE}_k = E\left[(\boldsymbol{r}^\top \boldsymbol{c}_k - b_k)^2\right] = \boldsymbol{c}_k^\top \boldsymbol{B} \boldsymbol{c}_k - 2\sqrt{p_k}\boldsymbol{c}_k^\top \boldsymbol{s}_k + 1 \tag{10}$$

where $\boldsymbol{B} = \sum_{j=1}^{N} p_j \boldsymbol{s}_j \boldsymbol{s}_j^\top + \sigma^2 \boldsymbol{I}_L = \boldsymbol{SPS}^\top + \sigma^2 \boldsymbol{I}_L$. The MMSE filter for the kth user is

$$\boldsymbol{c}_k = \arg\min_{\boldsymbol{c}_k} \mathrm{MSE}_k = \sqrt{p_i}\boldsymbol{B}^{-1}\boldsymbol{s}_k \tag{11}$$

Note that the MMSE solution can also be written as

$$\boldsymbol{c}_k = \frac{\sqrt{p_k}\left(\boldsymbol{A}_k + \sigma^2 \boldsymbol{I}_L\right)^{-1} \boldsymbol{s}_k}{1 + p_k \boldsymbol{s}_k^\top \left(\boldsymbol{A}_k + \sigma^2 \boldsymbol{I}_L\right)^{-1} \boldsymbol{s}_k} \tag{12}$$

where $\boldsymbol{A}_k = \sum_{j \neq k} p_j \boldsymbol{s}_j \boldsymbol{s}_j^\top$.

It was found in [3] that the network capacity of a CDMA system with MMSE receivers satisfies

$$N < L\left(1 + \frac{1}{\beta}\right) \tag{13}$$

It was shown in [3] that the network capacity with MMSE receivers is maximized if the signature sequence set is chosen to satisfy (8) if $N \leq L$ and (9) if $N > L$, and if the received powers of the users are chosen to be the same.

It was also shown in [3] that when the matched filters are used as the receivers, the network capacity is again given by (13) and the maximum is achieved by equal powers and signature sequence sets satisfying (8) and (9) for $N \leq L$ and $N > L$, respectively. As was observed in [3], this result is expected because once the signature set is chosen to satisfy (8) and (9), for $N \leq L$ and $N > L$, respectively, the MMSE receivers are scaled matched filters.

Although it is known that the sequences which satisfy (8) for $N \leq L$ and (9) for $N > L$ maximize the capacity of a single cell CDMA system, no simple algorithmic scheme is known to construct these sequence sets for general N and L. Note that when $N \leq L$, a simple Gram-Schmidt orthogonalization procedure would yield N orthogonal vectors starting with N linearly independent vectors. For $N > L$, no such simple construction scheme exists.

2 WBE Sequences and a Simple Observation

Welch developed lower bounds for the $2k$th power of the maximum correlation among a set of N unit energy sequences [6]. These lower bounds on the $2k$th powers of the maximum correlation were actually obtained from a lower bound on the sum of the $2k$th powers of the correlations of a sequence set.

$$\sum_{i=1}^{N}\sum_{j=1}^{N}(\boldsymbol{s}_i^\top \boldsymbol{s}_j)^{2k} \geq \frac{N^2}{\binom{L+k-1}{k}} \tag{14}$$

When $k = 1$, Welch's bound reduces to

$$\sum_{i=1}^{N}\sum_{j=1}^{N}(\boldsymbol{s}_i^\top \boldsymbol{s}_j)^{2} \geq \frac{N^2}{L} \tag{15}$$

For a simpler derivation of the bound (15), see [7,8]. Note that sequence sets satisfying (9) satisfy the bound in (15) with equality.

Consider a set of vectors, $\boldsymbol{s}_i$, $i = 1, \cdots, N$ used as signature sequences by N users. Assume that the signals of the users at the base station are received by matched filters and $p_i = p$, for all i. In this case, the mean squared error for the ith user is

$$\mathrm{MSE}_i = E\left[\left(\boldsymbol{r}^\top \boldsymbol{s}_i - b_i\right)^2\right] = \boldsymbol{s}_i^\top \boldsymbol{B} \boldsymbol{s}_i - 2\sqrt{p}\boldsymbol{s}_i^\top \boldsymbol{s}_i + 1 \tag{16}$$

where $\boldsymbol{r}$ is the received signal given in (2) and $\boldsymbol{B} = p\sum_{j=1}^{N} \boldsymbol{s}_j \boldsymbol{s}_j^\top + \sigma^2 \boldsymbol{I}_L$. The total mean squared error in the system is

$$\begin{aligned} \mathrm{MSE} = \sum_{i=1}^{N} \mathrm{MSE}_i &= \sum_{i=1}^{N} \boldsymbol{s}_i^\top \left(p \sum_{j=1}^{N} \boldsymbol{s}_j \boldsymbol{s}_j^\top + \sigma^2 \boldsymbol{I}_L\right) \boldsymbol{s}_i - 2\sqrt{p} \sum_{i=1}^{N} \boldsymbol{s}_i^\top \boldsymbol{s}_i + N \\ &= p \sum_{i=1}^{N} \sum_{j=1}^{N} (\boldsymbol{s}_i^\top \boldsymbol{s}_j)^2 - (2\sqrt{p} - \sigma^2) \sum_{i} \boldsymbol{s}_i^\top \boldsymbol{s}_i + N \end{aligned} \tag{17}$$

Defining total squared correlation (TSC) as

$$\mathrm{TSC} = \sum_{i=1}^{N} \sum_{j=1}^{N} (\boldsymbol{s}_i^\top \boldsymbol{s}_j)^2 \tag{18}$$

we observe that

$$\mathrm{MSE} = p\mathrm{TSC} - (2\sqrt{p} - \sigma^2) \sum_{i=1}^{N} \boldsymbol{s}_i^\top \boldsymbol{s}_i + N \tag{19}$$

Since the signature sequences are restricted to be of unit energy,

$$\mathrm{MSE} = p\mathrm{TSC} + (1 + \sigma^2 - 2\sqrt{p})N \tag{20}$$

and minimizing TSC subject to $\boldsymbol{s}_i^\top \boldsymbol{s}_i = 1$ is equivalent to minimizing MSE subject to the same condition. Since the sequence sets satisfying (8) for $N \leq L$, and (9) for $N > L$ minimize the TSC, they minimize the MSE as well. In other words, orthogonal sequences for $N \leq L$ and WBE sequences for $N > L$ are the *global optimal* solutions of the following two equivalent problems

$$\begin{array}{ll} \min & \mathrm{TSC} \\ \text{s. t.} & \boldsymbol{s}_i^\top \boldsymbol{s}_i = 1 \quad i = 1, \ldots, N \end{array} \qquad\qquad \begin{array}{ll} \min & \mathrm{MSE} \\ \text{s. t.} & \boldsymbol{s}_i^\top \boldsymbol{s}_i = 1 \quad i = 1, \ldots, N \end{array} \tag{21}$$

In the following we will show that if we replace any one of the N sequences in a set with the normalized MMSE filter corresponding to that sequence, the resulting set of sequences will have a smaller TSC. This simple observation will constitute the basis for the iterative algorithm we will propose in the next section.

We first separate the terms that depend on a particular signature sequence, $\boldsymbol{s}_k$, in the TSC expression

$$\mathrm{TSC} = (\boldsymbol{s}_k^\top \boldsymbol{s}_k)^2 + 2\boldsymbol{s}_k^\top \left(\sum_{j \neq k} \boldsymbol{s}_j \boldsymbol{s}_j^\top\right) \boldsymbol{s}_k + \sum_{i \neq k} \sum_{j \neq k} (\boldsymbol{s}_i^\top \boldsymbol{s}_j)^2 \tag{22}$$

Since we will always restrict ourselves to unit energy signature sequences, i.e., $\boldsymbol{s}_k^\top \boldsymbol{s}_k = 1$, we can add $2a^2\boldsymbol{s}_k^\top\boldsymbol{s}_k$ and subtract $2a^2$ in the above expression to get

$$\mathrm{TSC} = (\boldsymbol{s}_k^\top\boldsymbol{s}_k)^2 + 2\boldsymbol{s}_k^\top\left(\sum_{j\neq k}\boldsymbol{s}_j\boldsymbol{s}_j^\top + a^2\boldsymbol{I}_L\right)\boldsymbol{s}_k + \sum_{i\neq k}\sum_{j\neq k}(\boldsymbol{s}_i^\top\boldsymbol{s}_j)^2 - 2a^2 \tag{23}$$

Let us replace the signature sequence of user k with the unit energy vector

$$\boldsymbol{c}_k = \frac{\left(\boldsymbol{A}_k + a^2\boldsymbol{I}_L\right)^{-1}\boldsymbol{s}_k}{\left[\boldsymbol{s}_k^\top\left(\boldsymbol{A}_k + a^2\boldsymbol{I}_L\right)^{-2}\boldsymbol{s}_k\right]^{1/2}} \tag{24}$$

where $\boldsymbol{A}_k = \sum_{j\neq k}\boldsymbol{s}_j\boldsymbol{s}_j^\top$. Thus we map the set of signature sequences $\boldsymbol{S}$ to a new set of signatures

$$\bar{\boldsymbol{S}} = [\boldsymbol{s}_1, \boldsymbol{s}_2, \cdots, \boldsymbol{s}_{k-1}, \boldsymbol{c}_k, \boldsymbol{s}_{k+1}, \cdots, \boldsymbol{s}_N] \tag{25}$$

Note that $\boldsymbol{c}_k$ is the normalized MMSE filter for user k. Also note that this MMSE filter is a generalized one; $\boldsymbol{c}_k$ is the normalized MMSE filter for user k in a CDMA system where all other users transmit with signature sequences $\boldsymbol{s}_j$ where $j \neq k$, all users have received powers $p_i = 1$ and the variance of the AWGN is a^2. It will be apparent in what follows that any generalized normalized MMSE filter will be as good as any other in terms of constructing optimum sequences.

The total squared correlation of set $\boldsymbol{S}$ is

$$\mathrm{TSC} = (\boldsymbol{s}_k^\top\boldsymbol{s}_k)^2 + 2\boldsymbol{s}_k^\top\left(\boldsymbol{A}_k + a^2\boldsymbol{I}_L\right)\boldsymbol{s}_k + C - 2a^2 \tag{26}$$

where $C = \sum_{i\neq k}\sum_{j\neq k}(\boldsymbol{s}_i^\top\boldsymbol{s}_j)^2$ represents the squared correlation terms that are not affected by the update: $\boldsymbol{s}_k \to \boldsymbol{c}_k$. Equations (23) and (24) imply that the modified signature set $\bar{\boldsymbol{S}}$ has total squared correlation

$$\overline{\mathrm{TSC}} = (\boldsymbol{c}_k^\top\boldsymbol{c}_k)^2 + 2\frac{\boldsymbol{s}_k^\top\left(\boldsymbol{A}_k + a^2\boldsymbol{I}_L\right)^{-1}\boldsymbol{s}_k}{\boldsymbol{s}_k^\top\left(\boldsymbol{A}_k + a^2\boldsymbol{I}_L\right)^{-2}\boldsymbol{s}_k} + C - 2a^2 \tag{27}$$

Note that by the nature of the mapping in (24), $\boldsymbol{c}_k^\top\boldsymbol{c}_k = \boldsymbol{s}_k^\top\boldsymbol{s}_k = 1$. The following theorem verifies that replacing a particular sequence with its normalized MMSE receiver cannot increase the total squared correlation of the set. In other words, in terms of the TSC (or MSE) criteria, $\bar{\boldsymbol{S}}$ is a *better* set of signature sequences than $\boldsymbol{S}$.

Theorem 1 *If any signature $\boldsymbol{s}_k$ is replaced with the normalized MMSE receiver for that vector, $\boldsymbol{c}_k$, the total squared correlation decreases: $\overline{TSC} \leq TSC$. The inequality is satisfied with equality, i.e., $\overline{TSC} = TSC$, iff $\boldsymbol{c}_k = \boldsymbol{s}_k$.*

Due to space limitations, the proof of Theorem 1 and all other proofs will be omitted here. They can be found in [9].

3 An Iterative Algorithm

We observed in the previous section that given a set of unit energy vectors, if any one of these vectors is replaced with the corresponding normalized MMSE vector, then the TSC of the set decreases. We start with N unit length vectors $\boldsymbol{S}(0) = [\boldsymbol{s}_1(0), \cdots, \boldsymbol{s}_N(0)]$ at time 0. At iteration $(n+1)$ the algorithm replaces the vectors $\boldsymbol{S}(n) = [\boldsymbol{s}_1(n), \cdots, \boldsymbol{s}_N(n)]$ with their corresponding normalized MMSE filters one by one, and yields $\boldsymbol{S}(n+1) = [\boldsymbol{s}_1(n+1), \cdots, \boldsymbol{s}_N(n+1)]$. A complete iteration includes N intermediate steps. At the

kth intermediate step in iteration $(n+1)$, the first $(k-1)$ vectors have already been updated and the current vector set is

$$\boldsymbol{S}_{k-1}(n+1) = [\boldsymbol{s}_1(n+1), \cdots, \boldsymbol{s}_{k-1}(n+1), \boldsymbol{s}_k(n), \boldsymbol{s}_{k+1}(n), \cdots, \boldsymbol{s}_N(n)] \tag{28}$$

The kth vector is then updated according to

$$\boldsymbol{s}_k(n+1) = \frac{\left(\boldsymbol{A}_k(n+1) + a^2 \boldsymbol{I}_L\right)^{-1} \boldsymbol{s}_k(n)}{\left[\boldsymbol{s}_k^\top(n) \left(\boldsymbol{A}_k(n+1) + a^2 \boldsymbol{I}_L\right)^{-2} \boldsymbol{s}_k(n)\right]^{1/2}} \tag{29}$$

to yield the vector set

$$\boldsymbol{S}_k(n+1) = [\boldsymbol{s}_1(n+1), \cdots \boldsymbol{s}_{k-1}(n+1), \boldsymbol{s}_k(n+1), \boldsymbol{s}_{k+1}(n), \cdots, \boldsymbol{s}_N(n)] \tag{30}$$

The matrix $\boldsymbol{A}_k(n+1)$ in (29) is given as

$$\boldsymbol{A}_k(n+1) = \sum_{j<k} \boldsymbol{s}_j(n+1)\boldsymbol{s}_j^\top(n+1) + \sum_{j>k} \boldsymbol{s}_j(n)\boldsymbol{s}_j^\top(n) \tag{31}$$

4 Convergence of the Proposed Algorithm

Let $\mathrm{TSC}_k(n)$ denote the TSC of the set $\boldsymbol{S}_k(n)$ after the kth intermediate step in iteration n. In addition, let $\mathrm{TSC}(n)$ denote the TSC at the end of iteration n for set $\boldsymbol{S}(n)$. As a consequence of Theorem 1 we have

$$\mathrm{TSC}_{k+1}(n) \leq \mathrm{TSC}_k(n) \qquad \text{for } k = 0, \cdots, N-1; \quad n \geq 0 \tag{32}$$

By recursive application of (32) we have

$$\mathrm{TSC}(n+1) \leq \mathrm{TSC}(n) \tag{33}$$

Note that for $N \leq L$, TSC is lower bounded by N, and in general (including the case $N > L$) TSC is lower bounded by N^2/L from Welch's bound. Therefore the monotonically decreasing sequence $\mathrm{TSC}(n)$ converges to a finite number. For $N \leq L$, if $\mathrm{TSC}(n)$ converges to N, then $\boldsymbol{S}(n)$ converges to a set of N orthogonal sequences, and for $N > L$ if $\mathrm{TSC}(n)$ converges to N^2/L then $\boldsymbol{S}(n)$ converges to a set of WBE sequences. Note that

$$\mathrm{TSC}(n+1) = \mathrm{TSC}_N(n+1) \leq \mathrm{TSC}_{N-1}(n+1) \leq \cdots \leq \mathrm{TSC}_1(n+1) \leq \mathrm{TSC}(n) \tag{34}$$

Thus, at the fixed point where $\mathrm{TSC}(n+1) = \mathrm{TSC}(n)$ we should have

$$\mathrm{TSC}(n+1) = \mathrm{TSC}_N(n+1) = \mathrm{TSC}_{N-1}(n+1) = \cdots = \mathrm{TSC}_1(n+1) = \mathrm{TSC}(n) \tag{35}$$

From Theorem 1, this occurs iff $\boldsymbol{s}_k(n+1) = \boldsymbol{s}_k(n)$ for all k. Thus at the fixed point we must have $\boldsymbol{S}(n+1) = \boldsymbol{S}(n)$.

Let us denote the fixed point set of vectors as $\boldsymbol{S} = [\boldsymbol{s}_1, \cdots, \boldsymbol{s}_N]$. Let $\boldsymbol{B} = \boldsymbol{S}\boldsymbol{S}^\top + a^2\boldsymbol{I}_L$. From two alternative notations for the MMSE operation in (11) and (12), we will choose to use that given in (11). The fixed point $\boldsymbol{S}$ satisfies

$$\boldsymbol{B}\boldsymbol{s}_i = \phi_i \boldsymbol{s}_i \qquad \text{for } i = 1, \cdots, N \tag{36}$$

In the following two subsections, we will investigate the properties of the fixed point for the cases $N \leq L$ and $N > L$ separately. We will identify the conditions under which we can guarantee that the fixed point is actually a set of orthogonal vectors for $N \leq L$ and a set of WBE sequences for $N > L$. We will need the following claim.

Lemma 1 *If the signature $\boldsymbol{s}_k$ is replaced by $\boldsymbol{c}_k$, at step k of iteration n, then*

(a) $\det(\boldsymbol{S}_{k+1}^\top(n)\boldsymbol{S}_{k+1}(n)) \geq \det(\boldsymbol{S}_k^\top(n)\boldsymbol{S}_k(n))$

(b) $\det(\boldsymbol{S}_{k+1}(n)\boldsymbol{S}_{k+1}^\top(n)) \geq \det(\boldsymbol{S}_k(n)\boldsymbol{S}_k^\top(n))$

Lemma 1(a) implies that if $\det\left(\boldsymbol{S}^\top(0)\boldsymbol{S}(0)\right) > 0$, i.e., $\boldsymbol{S}^\top(0)\boldsymbol{S}(0)$ is invertible, then at the fixed point $\det\left(\boldsymbol{S}^\top\boldsymbol{S}\right) > 0$, i.e., $\boldsymbol{S}^\top\boldsymbol{S}$ is invertible. Similarly, Lemma 1(b) says that if $\det\left(\boldsymbol{S}(0)\boldsymbol{S}^\top(0)\right) > 0$, i.e., $\boldsymbol{S}(0)\boldsymbol{S}^\top(0)$ is invertible, then at the fixed point, $\det\left(\boldsymbol{S}\boldsymbol{S}^\top\right) > 0$, i.e., $\boldsymbol{S}\boldsymbol{S}^\top$ is invertible.

4.1 Properties of the Fixed Point: $N \leq L$

Multiplying the equality in (36) for user i with $\boldsymbol{s}_j^\top$ from left we obtain

$$\boldsymbol{s}_j^\top \boldsymbol{B}\boldsymbol{s}_i = \phi_i \boldsymbol{s}_j^\top \boldsymbol{s}_i \qquad \text{for } i = 1, \cdots, N;\ j = 1, \cdots, N \tag{37}$$

Inserting $\boldsymbol{B} = \boldsymbol{S}\boldsymbol{S}^\top + a^2\boldsymbol{I}_L$ into (37) and writing it in a matrix form we get

$$\boldsymbol{S}^\top\left(\boldsymbol{S}\boldsymbol{S}^\top + a^2\boldsymbol{I}_L\right)\boldsymbol{S} = \boldsymbol{S}^\top\boldsymbol{S}\boldsymbol{\Phi} \tag{38}$$

where $\boldsymbol{\Phi}$ is an $N \times N$ diagonal matrix with ϕ_i as its ith diagonal element. Equation (38) can be written as

$$\left(\boldsymbol{S}^\top\boldsymbol{S}\right)^2 + a^2\boldsymbol{S}^\top\boldsymbol{S} = \boldsymbol{S}^\top\boldsymbol{S}\boldsymbol{\Phi} \tag{39}$$

If the algorithm is started with N linearly independent vectors $\boldsymbol{S}(0)$, then by Lemma 1(a), $\boldsymbol{S}^\top\boldsymbol{S}$ will be invertible. Multiplying both sides of (39) with $(\boldsymbol{S}^\top\boldsymbol{S})^{-1}$ yields

$$\boldsymbol{S}^\top\boldsymbol{S} + a^2\boldsymbol{I}_N = \boldsymbol{\Phi} \tag{40}$$

Thus, $\boldsymbol{S}^\top\boldsymbol{S}$ is a diagonal matrix. By the nature of the algorithm $\boldsymbol{s}_i^\top\boldsymbol{s}_i = 1$ for all i, and the diagonal elements of $\boldsymbol{S}^\top\boldsymbol{S}$ are all unity. Therefore, at the fixed point for $N \leq L$ we have $\boldsymbol{S}^\top\boldsymbol{S} = \boldsymbol{I}_N$. Note from (40) that at the fixed point all ϕ_i have the same value.

Therefore, for $N \leq L$, if the algorithm is started with N linearly independent signature sequences, the proposed algorithm converges to an orthonormal set of signature sequences.

4.2 Properties of the Fixed point: $N > L$

Multiplying (36) for user i with $\boldsymbol{s}_j^\top$ from left, and multiplying the same equation for user j with $\boldsymbol{s}_i^\top$ from left, we obtain the following two equations

$$\boldsymbol{s}_j^\top \boldsymbol{B}\boldsymbol{s}_i = \phi_i \boldsymbol{s}_j^\top \boldsymbol{s}_i \qquad \text{and} \qquad \boldsymbol{s}_i^\top \boldsymbol{B}\boldsymbol{s}_j = \phi_j \boldsymbol{s}_i^\top \boldsymbol{s}_j \tag{41}$$

Since $\boldsymbol{B}$ is symmetric, we must have

$$\phi_i \boldsymbol{s}_j^\top \boldsymbol{s}_i = \phi_j \boldsymbol{s}_i^\top \boldsymbol{s}_j \tag{42}$$

Thus, if $\boldsymbol{s}_j^\top\boldsymbol{s}_i \neq 0$, then $\phi_i = \phi_j$. If the signature sequence set at the fixed point is not split into two or more orthogonal subsets, then we are guaranteed to have $\phi_i = \phi$ for all

i. Note that some vectors may be orthogonal to each other but as long as we do not have an orthogonal splitting of the vector set into two or more, we can still conclude that all ϕ_i have the same value. As we will see below, we can guarantee that the signature sequence set does not get split into orthogonal subsets at the fixed point by imposing quite mild conditions on the initial set of signature sequences, $\boldsymbol{S}(0)$. Without going into the details of these conditions, let us assume that they are satisfied and $\phi_i = \phi$ for all i. Thus

$$\boldsymbol{s}_j^\top \boldsymbol{B} \boldsymbol{s}_i = \phi \boldsymbol{s}_j^\top \boldsymbol{s}_i \qquad \text{for } i = 1, \cdots, N;\ j = 1, \cdots, N \tag{43}$$

Similar to the case of $N \leq L$, inserting $\boldsymbol{B} = \boldsymbol{S}\boldsymbol{S}^\top + a^2 \boldsymbol{I}_L$ into (43) and writing it in a matrix form we get

$$\boldsymbol{S}^\top \left(\boldsymbol{S}\boldsymbol{S}^\top + a^2 \boldsymbol{I}_L \right) \boldsymbol{S} = \phi \boldsymbol{S}^\top \boldsymbol{S} \tag{44}$$

Multiplying both sides of (44) from left by $\boldsymbol{S}$ and from right by $\boldsymbol{S}^\top$ yields

$$\left(\boldsymbol{S}\boldsymbol{S}^\top \right)^3 + a^2 \left(\boldsymbol{S}\boldsymbol{S}^\top \right)^2 = \phi \left(\boldsymbol{S}\boldsymbol{S}^\top \right)^2 \tag{45}$$

By Lemma 1(b), $\boldsymbol{S}\boldsymbol{S}^\top$ is invertible if the rank of $\boldsymbol{S}(0)$ is L. In this case, multiplying both sides of (45) by $(\boldsymbol{S}\boldsymbol{S}^\top)^{-2}$ yields

$$\boldsymbol{S}\boldsymbol{S}^\top + a^2 \boldsymbol{I}_L = \phi \boldsymbol{I}_L \tag{46}$$

Thus, $\boldsymbol{S}\boldsymbol{S}^\top = \alpha \boldsymbol{I}_L$. Note that $\text{trace}(\boldsymbol{S}\boldsymbol{S}^\top) = \text{trace}(\boldsymbol{S}^\top \boldsymbol{S}) = N$, since $\boldsymbol{s}_i^\top \boldsymbol{s}_i = 1$ for all i. Since $\text{trace}(\alpha \boldsymbol{I}_L) = \alpha L$, we have $\alpha = N/L$ and $\boldsymbol{S}\boldsymbol{S}^\top = (N/L)\boldsymbol{I}_L$.

Recursive application of the following Lemma verifies that for $N > L$, a signature set which is not split into orthogonal subsets at time 0 cannot be split into orthogonal subsets at any time n, including the fixed point.

Lemma 2 *For $N > L$, $\boldsymbol{S}(n)$ has two or more orthogonal subsets iff $\boldsymbol{S}(n-1)$ has two or more orthogonal subsets.*

Therefore, for $N > L$, if the algorithm is started with N signature sequences, L of which are linearly independent, and if the initial set of vectors are chosen such that it does not include two or more orthogonal subsets then the proposed algorithm converges to a set of WBE signature sequences.

5 Optimum Orthogonal Partitioning

We have observed that for $N > L$, our proof of convergence to WBE sequences requires that the initial signatures not be partitioned into orthogonal subsets. When $\boldsymbol{S}(0)$ can be partitioned into orthogonal subsets $\boldsymbol{S}'(0)$ and $\boldsymbol{S}''(0)$, we can view the signal subsets as belonging to orthogonal systems. The MMSE iteration preserves orthogonality among systems. Although TSC in this case still decreases monotonically, the algorithm is not guaranteed to converge to a WBE set and the TSC does not decrease to N^2/L. Note that within each orthogonal system, the signatures converge to either an orthogonal or WBE signature set depending on the number of users and the rank of each set. Denoting the TSC achieved by not partitioning the available bandwidth by TSC_1 and the TSC achieved by bandwidth partitioning by TSC_2, we can show that $\text{TSC}_1 \leq \text{TSC}_2$ [10]. Thus, we can conclude that orthogonal partitioning is suboptimal in general. But under certain conditions $\text{TSC}_1 = \text{TSC}_2$ and an orthogonal splitting can actually be as good as non-splitting. Note that the processing gain L is proportional to the available bandwidth. For a fixed bandwidth W, we can either have a signature sequence set with processing gain

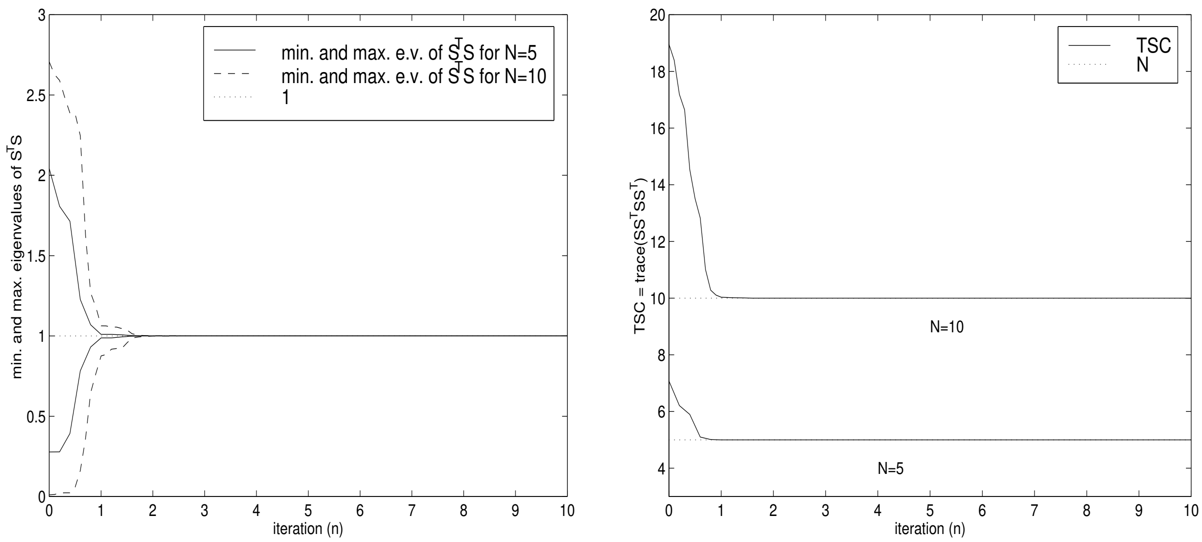

Figure 1: Minimum and maximum eigenvalues of matrix $\boldsymbol{S}^\top(n)\boldsymbol{S}(n)$, and TSC$(n)$.

L, or two orthogonal signature sequence sets with processing gains L_1 and $L_2 = L - L_1$ with N_1 and N_2 users, corresponding to bandwidth partitions W_1 and $W_2 = W - W_1$.

It can be shown [9,10] that only in two cases orthogonal partitioning of the signature sequence set yields equally good solutions as not partitioning: if $N \leq L$, any partitioning is as good as the optimal one as long as both spaces have enough dimensionality to assign orthogonal signature sequences to the users; and if $N > L$, we must have the equal loading condition $N_1/L_1 = N/L$.

6 Simulation Results

In this section we present some simple simulation results to verify our analysis. We take the processing gain to be $L = 10$. The initial signature sequences are created randomly. In all the figures, N updates take place between iterations n and $(n+1)$. In each update ith user's signature sequence is replaced with the corresponding normalized MMSE filter, for $i = 1, \cdots, N$. Figure 1 shows the minimum and maximum eigenvalues of the matrix $\boldsymbol{S}^\top(n)\boldsymbol{S}(n)$, and TSC$(n)$ as a function of the iteration index for number of users $N = 5, 10$. As expected, the minimum and maximum eigenvalues of $\boldsymbol{S}^\top(n)\boldsymbol{S}(n)$ converge to 1 implying that the matrix converges to $\boldsymbol{S}^\top\boldsymbol{S} = \boldsymbol{I}_N$, and the TSC converges to N since $N \leq L$.

Figure 2 shows the minimum and maximum eigenvalues of the matrix $\boldsymbol{S}(n)\boldsymbol{S}^\top(n)$ and TSC(n) for number of users $N = 20, 30, 40, 50$. As expected, the minimum and maximum eigenvalues of the matrix $\boldsymbol{S}(n)\boldsymbol{S}^\top(n)$ converge to N/L implying that the matrix converges to $\boldsymbol{S}\boldsymbol{S}^\top = (N/L)\boldsymbol{I}_L$, and the TSC converges to N^2/L, since in these cases $N > L$.

7 Conclusions

Recently it was shown that in order to maximize the capacity of a single cell synchronous CDMA system, if $N \leq L$ then N orthonormal sequences, and if $N > L$ then WBE sequences should be used as signature sequences. In this paper we proposed an algorithm which iteratively updates the signature sequences of the users one at a time and converges to the optimum signature sequence sets. The algorithm replaces one user's signature sequence with the normalized MMSE filter of that user at each update. Since the MMSE receiver can be constructed in a distributed fashion, the proposed algorithm can be

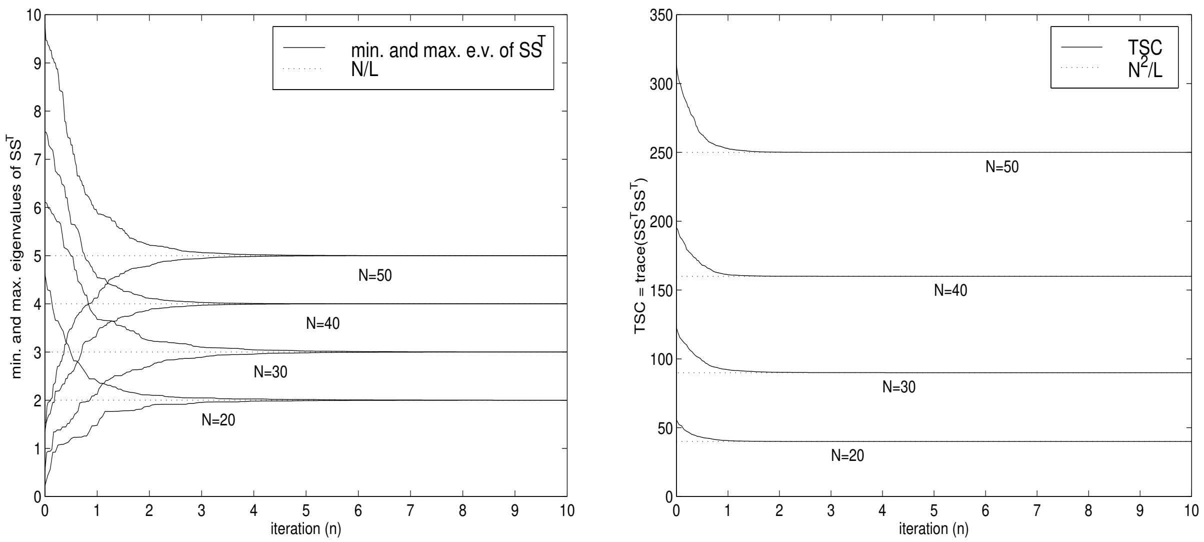

Figure 2: Minimum and maximum eigenvalues of matrix $\boldsymbol{S}^\top(n)\boldsymbol{S}(n)$, and $\mathrm{TSC}(n)$.

implemented distributedly. In this case, each user updates its own signature sequence by making some local measurements and the whole signature sequence set converges to an optimum one.

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