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Constellation Shaping, Nonlinear  
Precoding, and Trellis Coding for Voice-  
band Telephone Channel Modems



CONSTELLATION SHAPING,  
NONLINEAR PRECODING, AND  
TRELLIS CODING FOR VOICEBAND  
TELEPHONE CHANNEL MODEMS  
with Emphasis on ITU-T Recommenda-  
tion V.34

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## Preface

This book is essentially a history of the developments in signal constellation design, nonlinear pre-equalization, and trellis coding made by ITU-T Study Group XVII during its meetings to create a recommendation for a state-of-the-art voiceband telephone channel modem. It started deliberations in the fall of 1991 and by June of 1994 finalized the recommendation which was then given the V.34 label. The book is heavily based on notes I prepared for a series of lectures given to engineers at Penril Datability Networks (now the Signal Processing Group of Nortel Networks, Germantown, MD) during the ITU-T V.34 committee deliberations to keep them abreast of the latest technical proposals.

During the deliberations, the study group was called the V.fast committee and was jokingly called the V.last committee by members believing that no further improvements in voiceband modems would be made. The initial V.34 recommendation allowed data rates from 2400 to 28,800 bits per second in increments of 2400 bps. It was later amended to extend the maximum rate to 33,600 bps. More recently, the V.90 and V.92 recommendations were approved and allow rates up to 56,000 bps in the downstream direction, that is, from the server to the client modem, by taking advantage of the downstream digital network and PCM codec in the local office. The V.90 recommendation specifies using V.34 modulation in the upstream direction while V.92 allows PCM encoding in both the downstream and upstream directions in addition to V.34 modulation upstream.

The period during which the V.34 recommendation was being formulated was a very exciting time. New techniques were being proposed and discovered continually during the deliberations. Digital signal processor (DSP) technology was rapidly improving in terms of speed, smaller size, more internal RAM and ROM, and significantly reduced cost. The DSP technological advances allowed the committee to consider techniques that were significantly more com-

plex and effective than anything implemented before. And, of course, the political intrigues between individuals and companies made life interesting.

High speed quadrature amplitude modulation (QAM) modems use signal constellations with many points which are usually spaced on a regular grid for ease of implementation. These points are subsets of lattices. This book begins with a chapter introducing the basic theory and nomenclature for lattices as a reference for following chapters. V.34 modems use four-dimensional signal constellations for trellis coding and a constellation shaping technique called shell mapping over sixteen dimensions. Chapter 2 presents various performance measures for multidimensional constellations such as the constellation figure of merit, constellation expansion ratio, peak-to-average ratio, shaping gain, and coding gain. It is heavily based on the outstanding work by Forney and Wei [25]. Chapter 3 introduces the fundamentals of convolutional and trellis codes for reference. Early in the V.34 deliberations, Forney [23] proposed using trellis shaping to achieve shaping gain and this is discussed in Chapter 4. The inclusion of nonlinear precoding in the transmitter was then proposed. Chapter 5 presents the previously known Tomlinson-Harashima precoding technique and a new improved precoding method invented by Laroia [48] during the period of the V.34 committee meetings. Eyuboglu and Forney [18] then proposed combining trellis shaping and precoding into what they called trellis precoding and this is the subject of Chapter 6. Many committee participants felt that trellis precoding was too complex and AT&T proposed a technique for mapping input data bits to constellation points called a modulus encoder. A generalized version of the modulus encoder was included in the V.90 and V.92 recommendations. These modulus encoders are presented in Chapter 7. About this time, Laroia [45] proposed the use of shell mapping for assigning data bits to constellation points and simultaneously achieving shaping gain and it was included in the V.34 recommendation. Shell mapping is presented in Chapter 8. Early in the deliberations, the V.34 committee settled on using a four-dimensional signal constellation coupled with a four-dimensional trellis code. The details of this constellation are presented in Chapter 9. Near the end of the V.34 committee deliberations, an improved method for combining precoding with trellis coding in a feedback arrangement was discovered and accepted for inclusion in the recommendation. This is the subject of Chapter 10. Not long afterwards, the recommendation was finalized. Recommendation V.34 was the first V series recommendation to include a periodic sequence designed for fast equalizer training as a segment in its training sequence. This special periodic sequence, the optimum fractionally spaced equalizer, and a method for rapidly computing the equalizer coefficients by using FFT's is discussed in Chapter 11.

I would like to thank the highly talented engineers in the Penril V.34 data pump development team for encouraging me to give the series of lectures

on emerging V.34 modem technology. The team included Jalal Azim, Zoran Mladenovic, John Payton, Alek Purkovic, Dick Stuart, Dan Thomas, and Milan Vlanic. It was a pleasure to work with them. Also, we owe thanks to Rajiv Laroia for breaking the impasse between the AT&T modulus converter and Motorola (CODEX) trellis precoding camps by introducing shell mapping, an improved nonlinear precoding method, and the arrangement of the 4D trellis encoder and precoder in a feedback loop. The committee quickly accepted these new techniques over the modulus conversion and trellis precoding proposals. During the committee deliberations, Rajiv earned his PhD in Electrical Engineering at the University of Maryland and then went to work at the AT&T Bell Laboratories. This may be the first time a new PhD graduate has had his ideas included in an international standard!

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