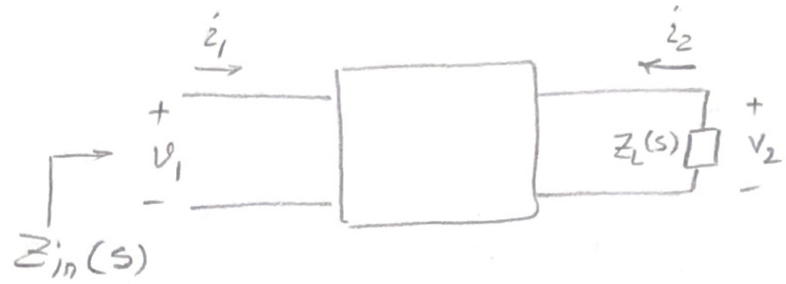
Homework # 5, problem 2 Fatemeh Alimardani

Negative Impedance Converters

a) $i_2 = K_i \, i_1$, $v_2 = K_v \, v_1$, K_i and K_v : non-zero current and voltage gain constants, load with $Z_L(s)$ and give $Z_{in}(s) = ?$

$$\begin{cases} v_1 = z_{11} \, i_1 + z_{12} \, i_2 \\ v_2 = z_{21} \, i_1 + z_{22} \, i_2 \end{cases}$$



$$Z_{in}(s) = \frac{v_1}{i_1} = ?$$

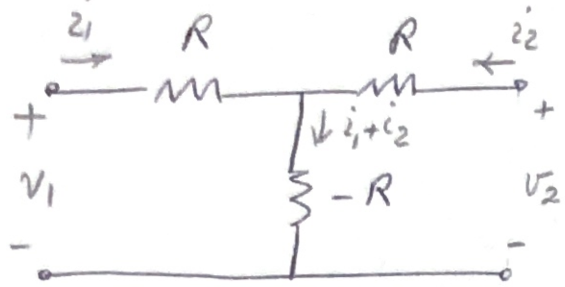
$v_2 = -Z_L \, i_2$ (1), $i_2 = K_i \, i_1$ (2), $v_2 = K_v \, v_1$ (3)

Plugging (2) and (3) in (1) :

$$K_v \, v_1 = -Z_L \, K_i \, i_1 \rightarrow \boxed{\frac{v_1}{i_1} = -\frac{K_i}{K_v} Z_L}$$

Negative impedance converter

b) R: positive #, Find Y, Z, S, Z_{in} when loaded with Z_L.



Finding Z:

$$\begin{cases} v_1 = z_{11} i_1 + z_{12} i_2 \\ v_2 = z_{21} i_1 + z_{22} i_2 \end{cases}, \quad Z = \begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix}$$

KVL equations:

$$\begin{cases} -v_1 + R i_1 - R(i_1+i_2) = 0 \rightarrow v_1 = -R i_2 \\ -v_2 + R i_2 - R(i_1+i_2) = 0 \rightarrow v_2 = -R i_1 \end{cases} \rightarrow Z = \begin{bmatrix} 0 & -R \\ -R & 0 \end{bmatrix}$$

Finding Y: (method 1)

$Y = Z^{-1}$ $\quad \det(Z) = 0 - (R^2) = -R^2 \rightarrow Y = \dfrac{1}{\det(Z)} \begin{bmatrix} 0 & R \\ R & 0 \end{bmatrix}$

$$\rightarrow Y = \begin{bmatrix} 0 & -1/R \\ -1/R & 0 \end{bmatrix}$$

(Method 1)

From abone KVL equations:

$$\begin{cases} i_2 = -\frac{1}{R} v_1 \\ i_1 = -\frac{1}{R} v_2 \end{cases}, \quad \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \underbrace{\begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix}}_{Y} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \rightarrow Y = \begin{bmatrix} 0 & -1/R \\ -1/R & 0 \end{bmatrix}$$

Finding S:

$$\begin{cases} 2v^i = v + Ri \\ 2v^r = v - Ri \end{cases}, \quad v^r = S v^i, \quad S: \text{Scattering Matrix}$$

$$S = (BR^{-1} + A)^{-1}(BR^{-1} - A), \quad A(S)v = B(S)i$$

For $R = 1_n$: $\quad S \overset{①}{=} (Z + 1_n)^{-1}(Z - 1_n) \quad$ or $\quad S \overset{②}{=} (1_n + Y)^{-1}(1_n - Y)$

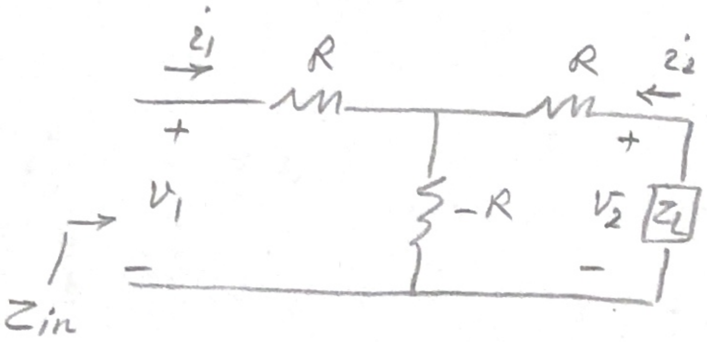
Using ① and $Z = \begin{bmatrix} 0 & -R \\ -R & 0 \end{bmatrix}$:

$$S = \left(\begin{bmatrix} 0 & -R \\ -R & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right)^{-1} \left(\begin{bmatrix} 0 & -R \\ -R & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right)$$

$$= \begin{bmatrix} 1 & -R \\ -R & 1 \end{bmatrix}^{-1} \begin{bmatrix} -1 & -R \\ -R & -1 \end{bmatrix} = \frac{1}{1-R^2} \begin{bmatrix} 1 & R \\ R & 1 \end{bmatrix} \begin{bmatrix} -1 & -R \\ -R & -1 \end{bmatrix} = \frac{1}{R^2 - 1} \begin{bmatrix} 1+R^2 & 2R^2 \\ 2R^2 & 1+R^2 \end{bmatrix}$$

Finding Z_{in} when loaded with Z_L:

$$Z_{in} = \frac{v_1}{i_1}$$



KVL: $\begin{cases} -v_1 + Ri_1 - R(i_1 + i_2) = 0 & ① \\ -v_2 + Ri_2 - R(i_1 + i_2) = 0 & ② \end{cases}$ and $v_2 = -Z_L i_2$ ③

②, ③ $\rightarrow \quad Z_L i_2 + Ri_2 - R(i_1 + i_2) = 0 \quad \rightarrow \quad Z_L i_2 = R i_1 \rightarrow i_2 = \frac{R}{Z_L} i_1$ ④

①, ④ $\quad -v_1 - Ri_2 = 0 \rightarrow -v_1 - R\left(\frac{R}{Z_L}\right) i_1 = 0 \rightarrow$

$$Z_{in} = \frac{v_1}{i_1} = \boxed{-\frac{R^2}{Z_L}}$$

c) Loaded with Z_L and $R=1$ ($K_i = K_v$), find S for Z_{in} and compare with that of Z_L $(n=1)$.

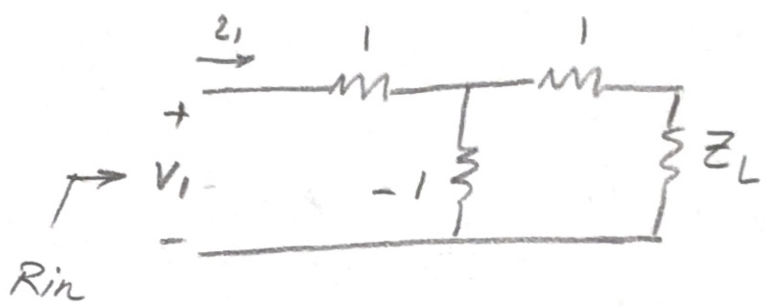
From part a)

$$Z_{in} = -\frac{K_i}{K_v} Z_L \xrightarrow{K_i = K_v} \boxed{Z_{in} = -Z_L}$$

From part b)

$$Z_{in} = -\frac{R^2}{Z_L} \xrightarrow{R = 1} \boxed{Z_{in} = -\frac{1}{R_L}}$$

1-port circuit :



$$R_{in} = 1 + \left(-1 \,//\, (1+Z_L)\right)$$

$$\rightarrow R_{in} = 1 + \frac{-(1+Z_L)}{Z_L} = \frac{Z_L - 1 - Z_L}{Z_L} = \frac{-1}{Z_L} = Z$$

$$S = (Z+1)^{-1}(Z-1) = \left(\frac{-1}{Z_L} + 1\right)^{-1}\left(-\frac{1}{Z_L} - 1\right) = \left(\frac{Z_L - 1}{Z_L}\right)^{-1}\left(\frac{-1-Z_L}{Z_L}\right)$$

$$= \left(\frac{Z_L}{Z_L - 1}\right)\left(\frac{-1-Z_L}{Z_L}\right) = \frac{1+Z_L}{1-Z_L}$$