Homework # 4 problem # 2

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a) Inverse Laplace Transform

$$F(s) = \frac{s^2 - as + b}{(s+a)(s^2+as+b)}, \quad a, b \in \mathbb{R} \qquad a: \text{non-negative}, \; b: \text{positive}$$

$$s^2 - as + b = M(s^2 + as + b) + (Ns + L)(s+a)$$

$$= Ms^2 + Mas + Mb + Ns^2 + Nsa + Ls + La$$

$$M + N = 1$$

$$Ma + Na + L = -a \longrightarrow (M+N)a + L = -a \longrightarrow L = -2a$$

$$Mb + La = b \longrightarrow Mb + (-2a)(a) = b \longrightarrow M = \frac{b + 2a^2}{b} = 1 + \frac{2a^2}{b}$$

$$\longrightarrow N = 1 - M = 1 - 1 - \frac{2a^2}{b} = -\frac{2a^2}{b}$$

$$\frac{s^2 - as + b}{(s+a)(s^2+as+b)} = \frac{1 + \frac{2a^2}{b}}{s+a} + \frac{-\frac{2a^2}{b}s - 2a}{\underbrace{s^2 + as + b}}$$

considering them real roots

$a^2 - 4b \geq 0$

$a^2 \geq 4b$

$$s^2 + as + b = 0 \longrightarrow s_{1,2} = \frac{-a \pm \sqrt{a^2 - 4b}}{2}$$

$$\frac{-\frac{2a^2}{b}s - 2a}{s^2 + as + b} = \frac{K}{(s - s_1)} + \frac{P}{(s - s_2)}$$

$$-\frac{2a^2}{b}s - 2a = K\left(s - \frac{-a - \sqrt{a^2 - 4b}}{2}\right) + P\left(s - \frac{-a + \sqrt{a^2 - 4b}}{2}\right)$$

I) $K + P = -\frac{2a^2}{b}$

$$-2a = K\left(\frac{a}{2} + \frac{\sqrt{a^2 + 4b}}{2}\right) + P\left(\frac{a}{2} - \frac{\sqrt{a^2 - 4b}}{2}\right)$$

$$-2a = \frac{a}{2}\underbrace{(K + P)}_{-\frac{2a^2}{b}} + \frac{K\sqrt{a^2 + 4b}}{2} - \frac{P\sqrt{a^2 - 4b}}{2}$$

$$-2a = -\frac{a^3}{b} + \frac{K\sqrt{a^2 + 4b}}{2} - \frac{P\sqrt{a^2 - 4b}}{2}$$

(II) $\frac{K\sqrt{a^2+4b}}{2} - \frac{P\sqrt{a^2-4b}}{2} = -2a + \frac{a^3}{b}$

Using (I) and (II), the values can be found using the Cramer's rule:

$$K = \frac{\begin{vmatrix} -2\frac{a^2}{b} & 1 \\ -2a+\frac{a^3}{b} & -\frac{\sqrt{a^2-4b}}{2} \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ \frac{\sqrt{a^2+4b}}{2} & -\frac{\sqrt{a^2-4b}}{2} \end{vmatrix}}, \quad P = \frac{\begin{vmatrix} 1 & -\frac{2a^2}{b} \\ \frac{\sqrt{a^2+4b}}{2} & -2a+\frac{a^3}{b} \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ \frac{\sqrt{a^2+4b}}{2} & -\frac{\sqrt{a^2-4b}}{2} \end{vmatrix}}$$

Having values of M, K, P and also S_1 & S_2 based on a & b, the partial fraction decomposition becomes:

$$\frac{S^2-aS+b}{(S+a)(S^2+aS+b)} = \underbrace{\frac{M}{S+a}}_{\substack{\downarrow \\ \text{ROC} \\ \text{Re}\,S \geq a}} + \underbrace{\frac{K}{S-S_1}}_{\substack{\downarrow \\ \text{ROC} \\ \text{Re}\,S \geq S_1}} + \underbrace{\frac{P}{S-S_2}}_{\substack{\downarrow \\ \text{ROC} \\ \text{Re}\,S \geq S_2}}$$

Region of convergence$_{total}$: $\wedge(ROC_1, ROC_2, ROC_3)$

Knowing that:

$$\mathcal{L}^{-1}\left\{\frac{1}{S+a}\right\} = e^{-at}u(t), \ \text{Re}\,S \geq -a$$

$$\mathcal{L}^{-1}\{F(s)\} = \left(Me^{-at} + Ke^{s_1 t} + Pe^{s_2 t}\right).u(t)$$

b) - Laplace transform: $F(s) = \int_0^{\infty} f(t)e^{-st}\,dt$, s: any complex number in ROC

- Fourier transform: $G(w) = \int_{-\infty}^{\infty} f(t)e^{-jwt}\,dt$, jw: lies on the imaginary axis

- If the imaginary axis lies in ROC of $\mathcal{L}\{F\}$, then $G(w) = F(jw)$. If not, then the Fourier transform doesn't exist.

$F(jw) = |F(jw)|e^{j\angle F(jw)}$ where $|F(jw)|$ is the magnitude & $\angle F(jw)$ is it's phase.

c) $Y(s) = F(s)\,U(s)$, $y(t) = ?$ when $a = 0$ & $u(t)$ = unit step function

Unit step function $u(t) = \begin{cases} 1 & t > 0 \\ 0 & t < 0 \end{cases}$, $U(s) = \frac{1}{s}$

$a = 0 \longrightarrow F(s) = \frac{s^2 + b}{s(s^2+b)} = \frac{1}{s} \rightarrow Y(s) = F(s).U(s) = \frac{1}{s} \cdot \frac{1}{s} = \frac{1}{s^2}$

$\mathcal{L}^{-1}\{Y(s)\} = y(t) = t$