ENEE 610 Fall 2019 Homework 4 Due Tu 10/01/19

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**hw4**

#1(50 points; scattering matrix)

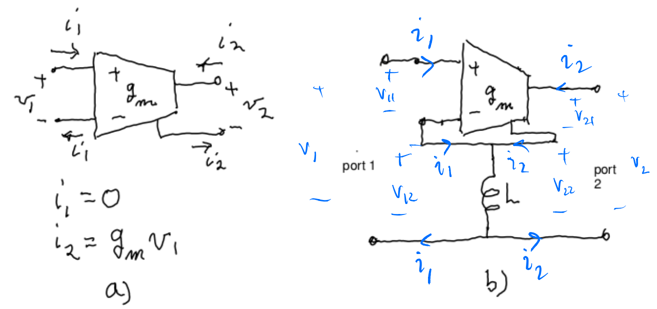
For the following circuit

a) Give the 2-port A(s)V=B(s)I description of the circuit of b) using the OTA of a) .

b) Give the scattering matrix for reference impedance $r1_2$ and for the case r=1

c) Interpret the $s_{21}$ in both cases including the position of poles and zeroes.

d) When r=1 for what $g_m$ and L is $s_{11}$ bounded-real




## Part a)

$$v_{11} = v_1 - v_{12} \quad \text{and} \quad v_{12} = sL i_2 \Rightarrow v_{11} = v_1 - sL i_2$$

$$i_2 = g_m v_{11} \quad \text{and} \quad v_{11} = v_1 - sL i_2 \Rightarrow i_2 = g_m (v_1 - sL i_2)$$

$$\Rightarrow (1 + sLg_m) i_2 = g_m v_1$$

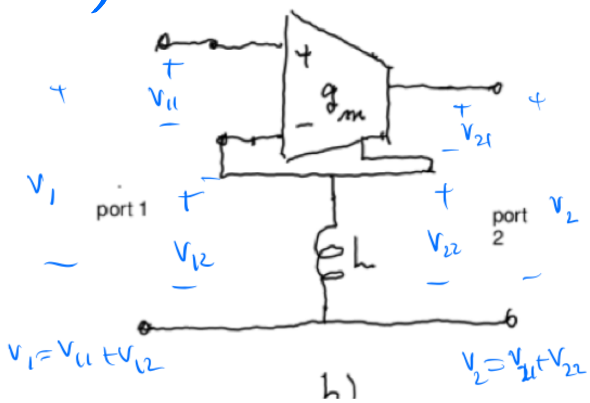
$$\Rightarrow i_2 = \frac{g_m}{1 + sLg_m} v_1$$

$$i_1 = 0 \quad \text{and} \quad i_2 = \frac{g_m}{1 + sLg_m} v_1$$

$$\Rightarrow \underbrace{\begin{bmatrix} 0 & 0 \\ \frac{g_m}{1+sLg_m} & 0 \end{bmatrix}}_{A(s)} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{B(s)} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix}$$

$$A(s) V = B(s) I$$

part b)




$$S = (BR^{-1} + A)^{-1}(BR^{-1} - A)$$

if the two ports of the 2-port are normalized to r

$$R = r\mathbb{1}_n = \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix} = r\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow R^{-1} = (1/r)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$S = ((1/r)B + A)^{-1}((1/r)B - A)$$

and for the case $r = 1$,

$$S = (B + A)^{-1}(B - A)$$

from part (a): $A(s) = \begin{bmatrix} 0 & 0 \\ \frac{g_m}{1+sLg_m} & 0 \end{bmatrix}$ ; $B(s) = \mathbb{1}_{2\times 2}$

$R = r\mathbb{1}_n$

$$S = ((1/r)\mathbb{1}_2 + A)^{-1}((1/r)\mathbb{1}_2 - A)$$

$$(1/r)\mathbb{1}_2 - A = \begin{bmatrix} 1/r & 0 \\ \frac{-g_m}{1+sLg_m} & 1/r \end{bmatrix} \; ; \; (1/r)\mathbb{1}_2 + A = \begin{bmatrix} 1/r & 0 \\ \frac{g_m}{1+sLg_m} & 1/r \end{bmatrix}$$

$$\det((1/r)\mathbb{1}_n + A) = \frac{1}{r^2} - \frac{g_m}{1+sLg_m} = \frac{sLg_m + (1 - r^2 g_m)}{r^2(1+sLg_m)}$$

$$\left((1/r)\mathbb{1}_2 + A\right)^{-1} = \frac{r^2(1+sLg_m)}{(1-r^2g_m)+sLg_m}\begin{bmatrix} 1/r & 0 \\ \frac{-g_m}{1+sLg_m} & 1/r \end{bmatrix}$$

$$S = \left((1/r)\mathbb{1}_2 + A\right)^{-1}\left((1/r)\mathbb{1}_2 - A\right)$$

$$= \frac{r^2(1+sLg_m)}{(1-r^2g_m)+sLg_m}\begin{bmatrix} 1/r & 0 \\ \frac{-g_m}{1+sLg_m} & 1/r \end{bmatrix}\begin{bmatrix} 1/r & 0 \\ \frac{-g_m}{1+sLg_m} & 1/r \end{bmatrix}$$

$$= \frac{r^2(1+sLg_m)}{(1-r^2g_m)+sLg_m}\begin{bmatrix} 1/r^2 & 0 \\ \frac{-2g_m}{r(1+sLg_m)} & 1/r^2 \end{bmatrix}$$

for $r = 1_\Omega \Rightarrow R = \mathbb{1}_2$

$$S = \frac{1+sLg_m}{(1-g_m)+sLg_m}\begin{bmatrix} 1 & 0 \\ \frac{-2g_m}{1+sLg_m} & 1 \end{bmatrix}$$

part 2) for $R = r \cdot 1/2$ and for $\underline{r = 1}$ $(R = 1/2)$

$$S_{21} = \frac{-2 g_m r}{(1 - r^2 g_m) + sL g_m}$$

$$S_{21} = \frac{-2 g_m}{(1 - g_m) + sL g_m}$$

one pole @

$$s = \frac{r^2 g_m - 1}{L g_m} = \frac{1}{L}\left(r^2 - \frac{1}{g_m}\right)$$

one pole @

$$s = \frac{g_m - 1}{L g_m} = \frac{1}{L}\left(1 - \frac{1}{g_m}\right)$$

One zero @ $s = \text{inf}.$

one zero @ $s = \text{inf}.$

Interpretation:

$S_{21}$ is reflected at port (2) and incident at port (1)

$$S_{21} = \frac{\text{Normalized reflected wave at port(2)}}{\text{Normalized incident wave at port(1)}}$$

part d) $r=1$

$$S_{11} = \frac{1+sLg_m}{(1-g_m)+sLg_m} \Rightarrow S_{11}(j\omega) = \frac{1+j\omega Lg_m}{(1-g_m)+j\omega Lg_m}$$

Rational Bounded Real (BR)

1) $S(s)$ has real coeff.'s $\qquad s=\sigma+j\omega$

2) $S(s)$ has no poles in $\sigma \geq 0$.
(analytic in closed RHP)

3) $\mathbb{1}_n - \underbrace{S^T(-j\omega)}_{S^{T*}(j\omega)=S^T(-j\omega)} S(j\omega) \geq 0$ (positive semi-def.)

(1) $L$ and $g_m$ need to be real

(2) no poles on RHP

one pole at $s = \frac{g_m-1}{Lg_m}$

$$\frac{g_m-1}{Lg_m} < 0 \Rightarrow g_m - 1 < Lg_m$$

$$g_m(1-L) < 1$$

$$\boxed{g_m < \frac{1}{1-L}}$$

(3) $$1 - \left(\frac{1-j\omega Lg_m}{(1-g_m)+j\omega Lg_m}\right)\left(\frac{1+j\omega Lg_m}{(1-g_m)-j\omega Lg_m}\right)$$

$$= 1 - \frac{1+(\omega Lg_m)^2}{(1-g_m)^2+(\omega Lg_m)^2} \geq 0$$

$$1 + \cancel{(\omega Lg_m)^2} \leq (1-g_m)^2 + \cancel{(\omega Lg_m)^2} \Rightarrow 1 \leq (1-g_m)^2$$

$$-1 \leq 1 - g_m \leq 1$$

$$-2 \leq -g_m \leq 0$$

$$\boxed{2 \geq g_m \geq 0}$$

#2 (50 points, biateral-Laplace transform).

a) Find the inverses, f(t), of the bilateral Laplace transform F(s)=[s^2-as+b]/{(s+a)(s^2+as+b)} for real non-negative a and positive b. Do this for all possible regions of convergence in the complex variable s-plane.

b) When s=jω the bilateral Laplace transform is the Fourier transform. When a>0 give the magnitude and phase for s=jω and discuss what this means about the region of convergence.

c) If U(s) is an input with Y(s) an output and Y(s)=F(s)U(s) give the possible outputs y(t) when a=0 and u(t) is the unit step function, u(t)=

$$F(s) = \frac{s^2 - as + b}{(s+a)(s^2+as+b)} \qquad a, b \in \mathbb{R}^+$$

$$F(s) = \frac{s^2 - as + b}{(s^2+as+b)(s+a)} = \frac{A}{s+a} + \frac{Bs+C}{s^2+as+b}$$

$$As^2 + Aas + Ab + Bs^2 + Cs + Bas + Ca$$

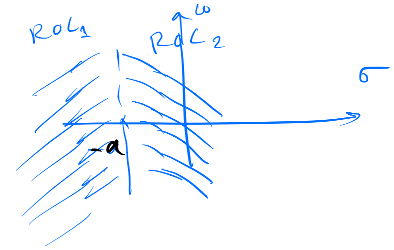
$$(A+B)s^2 + (Aa+Ba+C)s + (Ab+Ca) = s^2 - as + b$$

$$\begin{cases} A+B = 1 \Rightarrow A = 1-B & (\mathrm{I}) \\ Aa+Ba+C = -a \Rightarrow C = -a(A+B+1) \xRightarrow{(\mathrm{I})} \boxed{C = -2a} & (\mathrm{II}) \\ Ab + Ca = b \xRightarrow{C=-2a} Ab - 2a^2 = b \Rightarrow \boxed{A = 2a^2/b + 1} \\ \xRightarrow{(\mathrm{I})} \boxed{B = -2a^2/b} \end{cases}$$

$$F(s) = \frac{\overset{K_1}{2a^2/b + 1}}{s+a} + \frac{\overset{K_2}{-2a^2/b}\, s \; \overset{K_3}{-2a}}{s^2+as+b} = \frac{K_1}{s+a} + \frac{K_2 s}{s^2+as+b} + \frac{K_3}{s^2+as+b}$$

$$F(s) = \overbrace{\frac{(2a^2/b + 1)}{s+a}}^{F_1(s)} + \overbrace{\frac{(-2a^2/b\, s)}{s^2+as+b} + \frac{(-2a)}{s^2+as+b}}^{F_2(s)}$$

Note: $\mathcal{L}^{-1}\left\{\frac{1}{s+a}\right\} = \begin{cases} -e^{-at}u(-t) & \text{for ROC}_1 \quad Re(s) < -a \\ e^{-at}u(t) & \text{for ROC}_2 \quad Re(s) > a \end{cases}$ poles @ $s=-a$.




$$\mathcal{L}^{-1}\{F_1(s)\} = K_1 \mathcal{L}^{-1}\left\{\frac{1}{s+a}\right\} = \begin{cases} -K_1 e^{-at}u(-t) & : Re(s) < -a \\ K_1 e^{-at}u(t) & : Re(s) > a \end{cases} \; ; \; K_1 = 2a^2/b + 1$$

$$= \begin{cases} -(2a^2/b+1)e^{-at}u(-t) & Re(s) < -a \\ (2a^2/b+1)e^{-at}u(t) & Re(s) > a \end{cases}$$

poles at $p_{1,2} = -a/2 \pm \frac{\sqrt{a^2-4b}}{2}$

$s^2+as+b$

I. has 2 real roots $(a^2-4b>0)$ $p_1$ and $p_2$ $\quad -a < p_1 < -a/2$, $\; -a/2 < p_2 < 0$

$* \; a^2-4b>0 \Rightarrow a^2 > 4b \Rightarrow a > 2\sqrt{b}$

II. has a repeated root $(a^2-4b=0)$ $\quad p_1 = p_2 = -a/2$

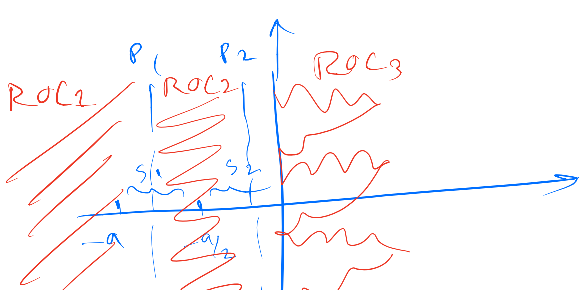
$a^2-4b=0 \Rightarrow a^2 = 4b \Rightarrow a = 2\sqrt{b}$

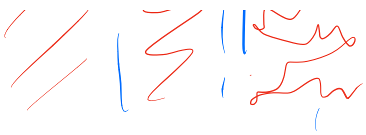
III. has a pair of conjugate complex roots $p_1$ and $p_2$

$a^2-4b<0 \Rightarrow a^2 < 4b \Rightarrow a < 2\sqrt{b}$

I. 2 real roots @ $p_1$ and $p_2$ $\quad \boxed{-a < p_1 < -a/2 \; ; \; -a/2 < p_2 < 0}$

$$\frac{k_2 s - k_3}{s^2+as+b} = \frac{k_2 s - k_3}{(s-p_1)(s-p_2)} = \frac{l_1}{s-p_1} + \frac{l_2}{s+p_2} \; ; \quad l_1 = \frac{k_2 p_1 - k_3}{p_1 - p_2}, \quad l_2 = \frac{k_2 p_2 - k_3}{p_2 - p_1}$$

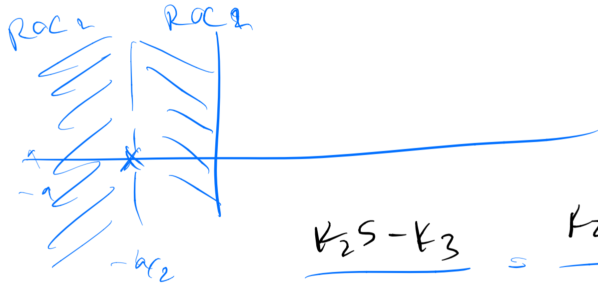




$$\mathcal{L}^{-1}\{F_2(s)\} = \begin{cases} -l_1 e^{-p_1 t} u(-t) - l_2 e^{-p_2 t} u(-t) & Re(s) < Re(p_1) \\ l_1 e^{-p_1 t} u(t) - l_2 e^{-p_2 t} u(t) & Re(p_1) < Re(s) < Re(p_2) \\ l_1 e^{-p_1 t} u(t) - l_2 e^{-p_2 t} u(-t) & Re(s) > Re(p_2) \end{cases}$$

if $s^2+as+b$ has a reapeated root $(a^2-4b=0)$ and $s_{1,2}=-a/2$

$s_1=s_2=-a/2=$




$$\frac{K_2 s - K_3}{s^2+as+b} = \frac{K_2 s + K_3}{(s+a/2)^2} = \frac{K_2 s}{(s+a/2)^2} + \frac{K_3}{(s+a/2)^2}$$

$$\mathcal{L}^{-1}\{F_2(s)\} = ? \quad \begin{cases} K_2 = -2a^2/b \\ K_3 = -2a \end{cases}$$

$$F_2(s) = \frac{-2a^2/b\, s - 2a}{(s+a/2)^2} = \frac{-2a^2/b\,(s+a/2) + (\frac{a^3}{b} - 2a)}{(s+a/2)^2}$$

$$= \frac{-2a^2/b\,(s+a/2)}{(s+a/2)^2} + \frac{(a^3/b - 2a)}{(s+a/2)^2}$$

$$= \frac{-2a^2/b}{(s+a/2)} + \frac{a^3/b - 2a}{(s+a/2)^2}$$

$$f_2(t) = \begin{cases} -2a^2/b\, e^{-a/2 t} u(t) + (a^3/b - 2a)\, t e^{-a/2 t} u(t) & \mathrm{Re}\, s > a/2 \\ 2a^2/b\, e^{-a/2 t} u(-t) - (a^3/b - 2a)\, t e^{-a/2 t} u(-t) & \mathrm{Re}\, s < a/2 \end{cases}$$

if $s^2+as+b$ has complex conjugat pair roots $(a^2-4b<0)$


$$\begin{cases} K_2 = -2a^2/b \\ K_3 = -2a \end{cases}$$

$$F_2(s) = \frac{K_2 s - K_3}{s^2+as+b} = \frac{K_2 s + K_3}{(s+a/2)^2+\left(\frac{\sqrt{a^2-4b}}{2}\right)^2} = \frac{-2a^2/b\,(s+a/2)+\left(\frac{a^3}{b}-2a\right)}{(s+a/2)^2+\left(\frac{\sqrt{a^2-4b}}{2}\right)^2}$$

$$= \frac{-2a^2/b\,(s+a/2)}{(s+a/2)^2+\left(\frac{\sqrt{a^2-4b}}{2}\right)^2} + \frac{2}{\sqrt{a^2-4b}}\,\frac{(a^3/b-2a)\frac{\sqrt{a^2-4b}}{2}}{(s+a/2)^2+\left(\frac{\sqrt{a^2-4b}}{2}\right)^2}$$

$$f_2(t) = \begin{cases} \left(-\frac{2a^2}{b}e^{-a/2\,t}\cos\left(\frac{\sqrt{a^2-4b}}{2}t\right) + \left(\frac{(a^3/b-2a)2}{\sqrt{a^2-4b}}\right)e^{-a/2\,t}\sin\frac{\sqrt{a^2-4b}}{2}t\right)u(t) \\ \left(\frac{2a^2}{b}e^{-a/2\,t}\cos\left(\frac{\sqrt{a^2-4b}}{2}t\right) - \left(\frac{(a^3/b-2a)2}{\sqrt{a^2-4b}}\right)e^{-a/2\,t}\sin\frac{\sqrt{a^2-4b}}{2}t\right)u(-t) \end{cases}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+as+b}\right\} = \begin{cases} ROC_1 \quad \frac{2}{\sqrt{a^2-4b}}e^{-a/2\,t}\sin\left(\frac{\sqrt{a^2-4b}}{2}t\right)u(t) \\ ROC_2\text{:} \quad \frac{2}{\sqrt{a^2-4b}}e^{-at}\sin\left(\frac{\sqrt{a^2-4b}}{2}t\right)u(-t) \end{cases}$$

b)

$$F(s) = \frac{s^2 - as + b}{(s+a)(s^2+as+b)} = \frac{s^2 - as + b}{s^3 + 2as^2 + (b+a^2)s + ab}$$

$$s^3 + as^2 + bs + as^2 + a^2 s + ab = s^3 + 2as^2 + (b+a^2)s + ab$$

$$F(j\omega) = \frac{(b-\omega^2) - j\omega a}{a(b-2\omega^2) + j((a^2+b)\omega - \omega^3)} = \frac{N(j\omega)}{D(j\omega)}$$

mag$(F(j\omega))$

$$|F(j\omega)| = \frac{|N(j\omega)|}{|D(j\omega)|} = \frac{\left((b-\omega^2)^2 + (-\omega a)^2\right)^{1/2}}{\left(a^2(b-2\omega^2)^2 + \left((a^2+b)\omega - \omega^3\right)^2\right)^{1/2}}$$

Phase$(F(j\omega))$

$$\angle F(j\omega) = \angle N(j\omega) - \angle D(j\omega) =$$

$$= \tan^{-1}\frac{-\omega a}{b-\omega^2} - \tan^{-1}\frac{(a^2+b)\omega - \omega^3}{b-2\omega^2}$$

if Fourier transform exist it means the region of convergence includes the $j\omega$ axis.

c.

$u(t) \Rightarrow U(s) = 1/s \; ; Re(s) > 0$

unit-step function

$$F(s) = \frac{s^2 - as + b}{(s+a)(s^2+as+b)} \overset{a=0}{\Longrightarrow} F(s) = \frac{\cancel{s^2+b}}{s(\cancel{s^2+b})} = \frac{1}{s}$$

$$Y(s) = F(s) \cdot U(s) = \frac{1}{s} \cdot \frac{1}{s} = 1/s^2 \quad ; \; Re(s) > 0$$

$$y(t) = t\,u(t)$$