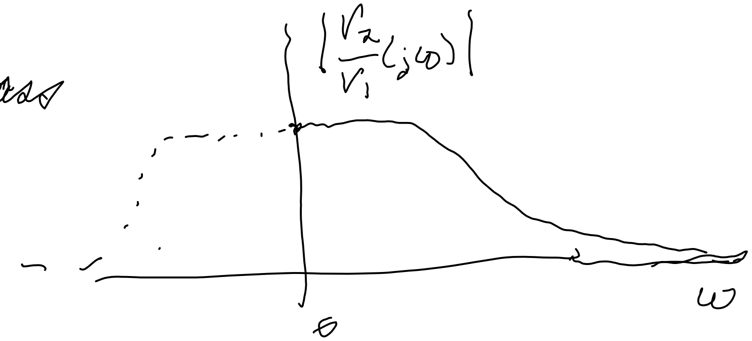
Note Title 10/29/2019

ENEE 610
10/29/19
addition 10/31/19

$$\frac{V_2}{V_1}(s) = \frac{1}{s^n + \dots + d_1 s + d_0}$$ low pass



high pass $s = 1/\hat{s}$ $s = 0 \Rightarrow \hat{s} = \infty$

$$\frac{V_2}{V_1}(\hat{s}) = \frac{\hat{s}^n}{d_0 \hat{s}^n + \dots + 1}$$

Maximally flat of low pass @ $\omega = 0$

$\Rightarrow$ as many derivatives $= 0$ as possible

$$\left|\frac{V_2}{V_1}(j\omega)\right|^2 = \frac{1}{D(j\omega)\,D(-j\omega)} = f(\omega^2)$$

$$\frac{d\left|\frac{V_2}{V_1}(j\omega)\right|}{d\omega} \; ; \quad \frac{d\left|\frac{V_2}{V_1}(j\omega)\right|^2}{d\omega} = 2\left|\frac{V_2}{V_1}(j\omega)\right| \frac{d\left|\frac{V_2}{V_1}(j\omega)\right|}{d\omega}$$

$$\frac{d\,1/f(\omega^2)}{d\omega} = -\frac{1}{f(\omega^2)}\frac{d f(\omega^2)}{d\omega} \qquad\qquad j^{2n}(-1)^n = (-1)^n(-1)^n$$

$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \shortparallel$$

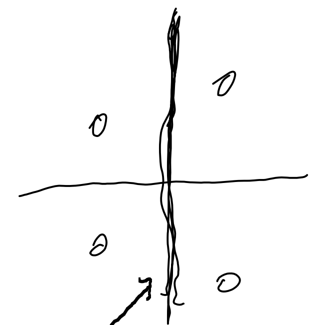
$$\frac{1}{f(\omega^2)} = \left((j\omega)^n + \cdots + 1\right)\left((-j\omega)^n + \cdots + 1\right) = (-1)^{2n}\,\omega^{2n} + \cdots + 1$$

$$\Rightarrow \text{Taylor series expansion of } \frac{1}{f(\omega^2)} = \left.\frac{1}{f}\right|_{\omega=0} + \alpha_1 \left.\frac{d\,1/f}{d\omega}\right|_{\omega=0} + \alpha_2 \left.\frac{d^2\,1/f}{d\omega^2}\right|_{\omega=0} + \cdots + (-1)^{2n+1}\omega^{2n}$$

$$\Rightarrow \frac{1}{\left|\frac{V_2}{V_1}(j\omega)\right|^2} = 1 + \cdots + (+1)^n\,\omega^{2n} = D(j\omega)\,D^*(j\omega) = |D(j\omega)|^2$$

$$D(j\omega)D(-j\omega)\Big|_{\omega = s/j} = 1 + (+1)^n \omega^{2n}$$

(analytic continuation to whole s-plane



given on $j\omega$ axis continue to whole s-plane by $\omega = s/j$

$$D(s)D(-s) = 1 + \left(\frac{s}{j}\right)^{2n} = 1 + (-1)^n s^{2n} = 0$$

$$(-1)^n s^{2n} = -1 \Rightarrow s^{2n} = (-1)^{n+1} = e^{j(\pi)(n+1) + j(2\pi \cdot k)}$$

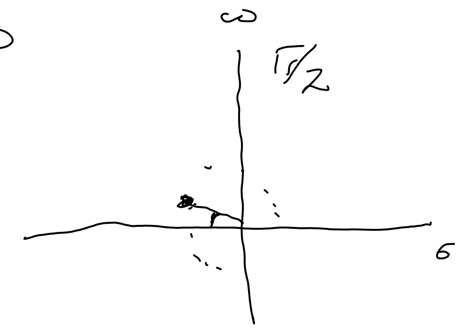
$$= e^{j(\pi(n+1) + 2k\pi)}$$

$k = 0, 1, 2, \ldots$

$$s_{o,k} = e^{j\left[\frac{(n+1)\pi}{2n} + \frac{2k\pi}{2n}\right]}$$

$k = 0, 1, 2, \ldots, 2n$

Ex: $n = 1$, max flat $\left|\frac{V_2}{V_1}(j\omega)\right|$ @ $\omega = 0$



$$s_{0,k=0} = e^{j\left(\frac{3\pi}{4}\right)+0}$$

$\measuredangle = \pi - \frac{3\pi}{4} = \frac{4\pi - 3\pi}{4} = \pi/4$

$$= -\cos \pi/4 + j \sin \pi/4$$

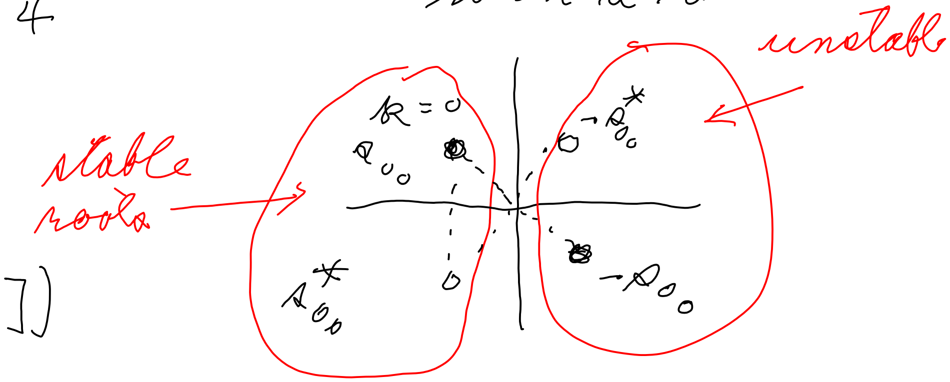
$\frac{\pi}{4} = \frac{180}{4} = 45^{\circ}$

$$= -\frac{\sqrt{2}}{2} + j\frac{\sqrt{2}}{2}$$

~~as~~ $D(s)D(-s) = 0$ then $-s$ is a root when s is

all roots of $D(s)D(-s)$ are these 4

$s = \pm \cos \pi/4 \pm j \sin \pi/4$



$$D(s) = \left(s - \left[-\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}}\right]\right)\left(s - \left[-\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}\right]\right)$$

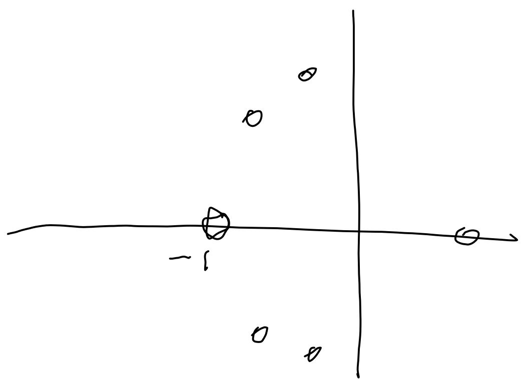
$$= (s - s_{00})(s - s_{00}^*)$$

$$= s^2 - (s_{00} + s_{00}^*)s + s_{00}s_{00}^* = s^2 - \left(\frac{-2}{\sqrt{2}}\right)s + \left(\sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2}\right)^2$$

$D(s) = s^2 + \sqrt{2}\, s + 1$; $\frac{V_2}{V_1}(s) = \frac{1}{s^2 + \sqrt{2}\, s + 1}$ is maximally flat magnitude at $\omega = 0$

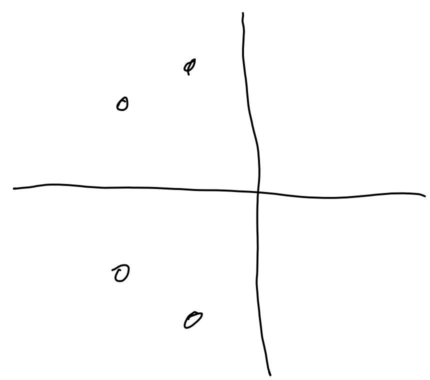
$=$ Butterworth polynomial degree 2

for n = odd



always an $s+1$ in $D(s)$ & 2nd order factors with complex roots of $|\text{root}| = 1$

for n even



low pass filters

$$\frac{1}{B_n(s)}$$

$B_n(s)$ = Butterworth polynomial

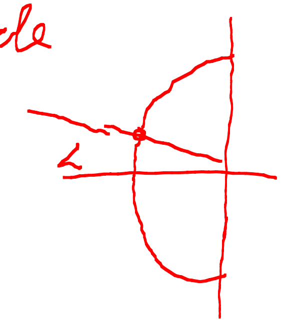
(tables exist)

added page

The polynomials can be found more explicity.

$e^{j\left(\frac{(n+1)\pi}{2n}+\frac{2k\pi}{2n}\right)}$ has all its roots evenly dividing the unit circle with those in the left plane at an angle from the x axis $\measuredangle_k = \pi - \left(\frac{\pi}{2}+\frac{\pi}{2n}+\frac{2k\pi}{2n}\right)$ $k=0,1,\dots,n-1$

$$= [n-(2k+1)]\frac{\pi}{2n}$$



∴ the x axis projection is

$$x_k = -\cos\left(\frac{n-(2k+1)}{2n}\pi\right) \quad k=0,\dots,n-1 \;; \quad s_k = x_k + jy_k = \text{zeros of } B_n(s) = 1 \text{ as on unit circle}$$

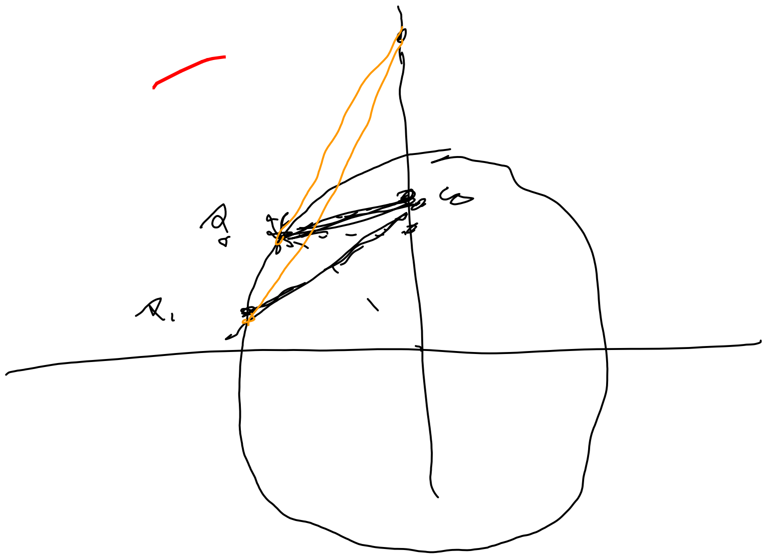
$$\therefore B_n(s) = \prod_{k=0}^{n-1}(s-s_k) = \begin{cases} (s+1)\prod_{k=0}^{(n-3)/2}(s-s_k)(s-s_k^*) & n=\text{odd} \quad \left(k=\frac{n-1}{2} \text{ gives } s+1\right) \\ \prod_{k=1}^{n/2}(s-s_k)(s-s_k^*) & n=\text{even} \end{cases}$$

$$= (s+1)\prod_{k=0}^{(n-3)/2} s^2-(s_k+s_k^*)s+s_ks_k^*$$

$$= \prod_{k=1}^{n/2} s^2-2x_k s+1$$

$$\therefore B_n(s) = \begin{cases} (s+1)\prod_{k=0}^{(n-3)/2}\left(s^2+2\cos\left(\frac{n-(2k+1)}{2n}\pi\right)s+1\right) & n \text{ odd} \\ \prod_{k=1}^{(n/2)}\left(s^2+2\cos\left(\frac{n-(2k+1)}{2n}\pi\right)s+1\right) & n \text{ even} \end{cases}$$

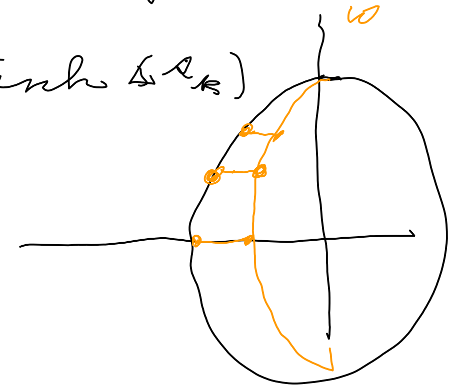
is the factored strictly Hurwitz Butterworth polynomial all these roots are on $|\alpha| = 1$ the unit circle



$$\frac{V_2}{V_1}(j\omega) = \frac{1}{(j\omega - \alpha_0)(j\omega - \alpha_1)\cdots} \Rightarrow \left|\frac{V_2}{V_1}\right| = \frac{1}{|j\omega - \alpha_0|} \cdot \frac{1}{|j\omega - \alpha_1|} \cdots$$

all these distances give the product = max flat if denominator is a Butterworth polynomial

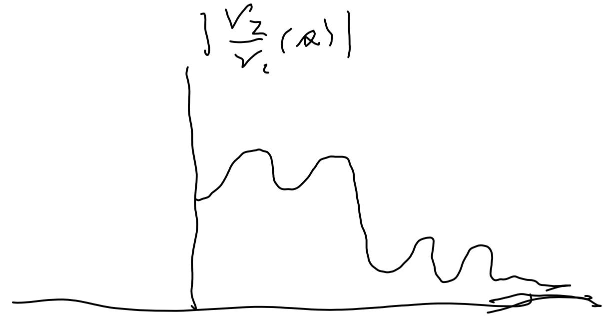
To get equal ripple put the zeroes of $D(s)$ on an ellipse at same y coordinate as for Butterworth polynomial multiplied by something like $(1-\epsilon^2 \tanh \Delta s_k)$



Use the Tschebyshev = Chebishev

y

for equal ripple pass & stop
$\Longrightarrow$ Elliptic filters



Books tabulate $\Rightarrow$ Skwerzinski

Use of adjoint for optimisation

Find a good approximation given a known $\frac{V_2}{V_1}(j\omega) = \hat{T}(j\omega)$

Need an error function of $\hat{T}(j\omega)$ vs what can get $T(j\omega)$

one possible $\mathcal{E}(T(j\omega)) = \int_0^\infty \underbrace{\left| T(j\omega) - \hat{T}(j\omega) \right|^2 d\omega}_{\text{for } L_2 \text{ functions}}$

desire $\mathcal{E}$ a minimum

Use a Taylor series expansion

$\mathcal{E}(T(j\omega)$, parameters of $T(j\omega) = x$ a vector of parameters like C, R, L, gyrator g's, etc.

$$\mathcal{E} = \mathcal{E}\Big|_{x_0} + \overset{\text{gradient}}{\nabla \mathcal{E}}\Big|_{x_0} (x - x_0) + \frac{1}{2!}(x - x_0)^T \underset{\text{2nd derivatives = Hessian}}{H} (x - x_0) + \dots$$