$$\frac{v_2}{v_1}(s) = \frac{n_3 s^3 + n_2 s^2 + n_1 s + n_0}{d_3 s^3 + d_2 s^2 + d_1 s + d_0} = D + \frac{\hat{n}_2 s^2 + \hat{n}_1 s + \hat{n}_0}{s^3 + d_2 s^2 + d_1 s + d_0}$$

$d_3 = 1$, $D = n_3/d_3 = n_3$

$\beta_1 = \hat{n}_0$, $\beta_2 = \hat{n}_1$, $\beta_3 = \hat{n}_2$

State equations

$$s x = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\alpha_1 & -\alpha_2 & -\alpha_3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} v_1$$

$\alpha_1 = d_0$, $\alpha_2 = d_1$, $\alpha_3 = d_2$

$$v_2 = [\beta_1 \ \beta_2 \ \beta_3] x + D v_1$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\frac{dx_1}{dt} = x_2$$

$$\frac{dx_2}{dt} = x_3 = \frac{d^2 x_1}{dt^2}$$

$$\dddot{x}_1 = \frac{dx_3}{dt} = -\alpha_1 x_1 - \alpha_2 x_2 - \alpha_3 x_3 + v_1 = -\alpha_1 x_1 - \alpha_2 \frac{dx_1}{dt} - \alpha_3 \ddot{x}_1 + v_1$$

$$\dddot{x}_1 + \alpha_3 \ddot{x}_1 + \alpha_2 \dot{x}_1 + \alpha_1 x_1 = v_1 \Rightarrow x_1 = \frac{v_1}{s^3 + \alpha_3 s^2 + \alpha_2 s + \alpha_1}$$

$$v_2 = \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + D v_1$$

$$= \beta_1 x_1 + \beta_2 s x_1 + \beta_3 s^2 x_1 + D v_1 = \frac{[\beta_1 + \beta_2 s + \beta_3 s^2]\, v_1}{s^3 + \alpha_3 s^2 + \alpha_2 s + \alpha_1} + D v_1$$

Compare with given v_2/v_1

$$s x = \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\alpha_0 & -\alpha_1 & -\alpha_2 \end{bmatrix}}_{\substack{\text{companion} \\ \text{matrix for } \frac{v_2}{v_1}(s)}} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} v_1 , \quad v_2 = [\hat{n}_0, \hat{n}_1, \hat{n}_2]\, x + D v_1$$

$$\Rightarrow \quad s x = A x + B u$$
$$y = C x + D u$$

If $u = 0$ then $\quad e^{At} x_0$ solves $dx/dt = Ax$, $x_0 =$ constant k-vector if $k = \dim$ of x

definition: $e^{At} = \sum_{i=0}^{\infty} \frac{A^i t^i}{i!}$; $\frac{de^{At}}{dt} = \sum_{i=1}^{\infty} \frac{A^i t^{i-1} \times (i)}{i!}$

$$= A \cdot \underbrace{\sum_{j=0}^{\infty} \frac{A^j t^j}{(i-1)! = j!}}$$

$$= A \cdot e^{At}$$

$\frac{dx}{dt} = Ax$

$\underbrace{x = e^{At} x_0, \quad x_0 = e^{0t} x_0}$

zero input solution, $-\infty < t < \infty$

can put on a unit step function, $1(t) = \begin{cases} 1 & \text{if } t > 0 \\ 0 & \text{if } t < 0 \end{cases}$

$x(t) = e^{At} x_0 1(t)$ for a physical solution & $x_0 = x(0)$

$$\frac{d}{dt}x = Ax + Bu \quad , \quad y = Cx + D$$

$$\mathcal{L}\left[\dot{x}\,1(t)\right] = \int_{-\infty}^{\infty} \frac{dx}{dt}\,1(t)\,e^{-\alpha t}\,dt$$

$$\frac{d(uv)}{dt}\cdot 1(t) = \frac{du}{dt}\,v\,1(t) + u\,\frac{dv}{dt}\,1(t)$$

$$\int_{-\infty}^{\infty} \frac{d(uv)}{dt}\,1(t)\,dt = \int_{-\infty}^{\infty} v\frac{du}{dt}\,1(t)\,dt + \int_{-\infty}^{\infty} u\frac{dv}{dt}\,1(t)\,dt$$

$$u = x$$
$$v = e^{-\alpha t} \Rightarrow \int_{-\infty}^{\infty} \frac{d\,xe^{-\alpha t}}{dt}\cdot 1(t)\,dt = \int_{-\infty \Rightarrow 0}^{\infty} e^{-\alpha t}\,\dot{x}\,1(t)\,dt + \int_{-\infty \Rightarrow 0}^{\infty} x(-\alpha)\,e^{-\alpha t}\,1(t)\,dt$$

$$x(t)e^{-\alpha t}\Big|_{0}^{\infty} = \mathcal{L}\left[\dot{x}\,1(t)\right] + (-\alpha)\,\mathcal{L}\left[x\,1(t)\right]$$

$$\mathcal{L}_u[\dot{x}] = s\,\mathcal{L}_u[x] - x(0)e^{-s \times 0}$$

in $\sigma > 0$

$$\frac{dx}{dt} = Ax + Bu \Rightarrow s\mathcal{L}[x] = A\mathcal{L}[x] + x(0) + B\mathcal{L}[u]$$

$$\mathcal{L}[x] = (s1_R - A)^{-1}\{B\mathcal{L}[u] + x(0)\}$$

$$= (s1_R - A)^{-1}\cdot\{B\mathcal{L}[u] + x(0)\mathcal{L}[\delta]\}$$

here $e^{At}x_0$ is the response $x(t)$ to $x_0\delta(t)$

$\Rightarrow e^{At}$ is the impulse response

$$x(t) = \mathcal{L}^{-1}\left[(s1_R - A)^{-1}\cdot\{B\cdot\mathcal{L}[u] + x_0\delta(t)\}\right]$$

$$x_0 = \mathcal{L}[x_0\delta(t)]$$

$$\int_{-\infty}^{\infty} x_0\,\delta(t)e^{-st}\,dt$$

$$= x_0\int_{-\infty}^{\infty}\delta(t)\,\underset{1}{\underset{\shortparallel}{e^{-s0}}}\,dt$$

$$= x_0\underbrace{\int_{-\infty}^{\infty}\delta(t)\,dt}_{\text{area} = 1}$$

$\delta(t) =$ unit impulse

$= d1(t)/dt$

from Laplace transform theory

$$\mathcal{L}[a(t)]\cdot\mathcal{L}[b(t)] = \mathcal{L}[a(t) * b(t)]$$

$$a(t) * b(t) = \int_{-\infty}^{\infty} a(t-\tau)b(\tau)\,d(\tau)$$

$$x(t) = e^{At} x_0 + \underbrace{\int_{-\infty}^{\infty} e^{A(t-\tau)} B \cdot u(\tau)\, d\tau}_{\text{convolution}} \qquad y(t) = C x(t) + D u(t)$$

state   impulse response   initial state   input

Ex:

$$\frac{v_2}{v_1} = \frac{s^2+3s+1}{(s+1)(s+2)(s+3)} = \frac{s^2+3s+1}{(s^2+3s+2)(s+3)} = \frac{s^2+3s+1}{s^3+6s^2+11s+6}$$

$$s\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6, & -11, & -6 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} v_1, \quad v_2 = [1, \ 3, \ 1]\, x$$

For e^{At} form $(s I_3 - A)^{-1} = \begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 6 & 11 & s+6 \end{bmatrix}^{-1}$

$$\det(\lambda I_3 - A) = \lambda \begin{vmatrix} \lambda & -1 \\ 11 & \lambda+6 \end{vmatrix} + 0 \cdot 1 \cdot 1 + 6 \begin{vmatrix} -1 & 0 \\ \lambda & -1 \end{vmatrix} = \lambda(\lambda^2 + 6\lambda + 11) + 6(1)$$

$$= \lambda^3 + 6\lambda^2 + 11\lambda + 6 = (\lambda+1)(\lambda+2)(\lambda+3)$$

$\Delta_{11} = \begin{vmatrix} \lambda & \\ & \lambda+6 \end{vmatrix} = \lambda^2 + 6\lambda + 11$, $\Delta_{21} = (-1)^{2+1} \begin{vmatrix} -1 & 0 \\ 11 & \lambda+6 \end{vmatrix} = +(\lambda+6)$, $\Delta_{31} = \begin{vmatrix} -1 & 0 \\ \lambda & -1 \end{vmatrix} = 1$

$\Delta_{12} = (-1)^{1+2} \begin{vmatrix} 0 & -1 \\ 6 & \lambda+6 \end{vmatrix} = -6$ $\quad \Delta_{22} = \begin{vmatrix} \lambda & 0 \\ 6 & \lambda+6 \end{vmatrix} = \lambda^2 + 6\lambda$ $\quad \Delta_{32} = (-1)^{2+3} \begin{vmatrix} \lambda & 0 \\ 0 & -1 \end{vmatrix} = \lambda$

$\Delta_{13} = \begin{vmatrix} 0 & \lambda \\ 6 & 11 \end{vmatrix} = -6\lambda$ $\quad \Delta_{23} = (-1) \begin{vmatrix} \lambda & -1 \\ 6 & 11 \end{vmatrix} = -11\lambda - 6$ $\quad \Delta_{33} = \begin{vmatrix} \lambda & -1 \\ 6 & \lambda \end{vmatrix} = \lambda^2$

$$(\lambda I_3 - A)^{-1} = \frac{1}{(\lambda+1)(\lambda+2)(\lambda+3)} \begin{bmatrix} \lambda^2+6\lambda+11 & \lambda+6 & 1 \\ -6 & \lambda^2+6\lambda & \lambda \\ -6\lambda & -11\lambda-6 & \lambda^2 \end{bmatrix}$$

$$= \frac{1}{\lambda+1} \begin{bmatrix} 3 & 5/2 & 1/2 \\ -3 & -5/2 & -1/2 \\ 3 & 5/2 & 1/2 \end{bmatrix} + \frac{1}{\lambda+2} \begin{bmatrix} -3 & -4 & -1 \\ 6 & 8 & 2 \\ -12 & -16 & -4 \end{bmatrix} + \frac{1}{\lambda+3} \begin{bmatrix} 1 & 3/2 & 1/2 \\ -3 & -9/2 & -3/2 \\ 9 & 13/2 & 9/2 \end{bmatrix}$$

$$\alpha_{11} = \frac{1}{(-1+2)(-1+3)}\left((-1)^2 - 6 + 11\right) = \frac{6}{2} = 3$$

$$e^{At} = e^{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6, & -11, & -6 \end{bmatrix}t} = \begin{bmatrix} 3 & 5/2 & 1/2 \\ -3 & -5/2 & -1/2 \\ 3 & 5/2 & 1/2 \end{bmatrix} e^{-t} + \begin{bmatrix} -3 & -4 & -1 \\ 6 & 8 & 2 \\ -12 & -16 & -4 \end{bmatrix} e^{-2t} + \begin{bmatrix} 1 & 3/2 & 1/2 \\ -3 & -9/2 & -3/2 \\ 9 & 13/2 & 9/2 \end{bmatrix} e^{-3t}$$

Using $\mathcal{L}[\cdot]$ avoids ∞ series for e^{At}.