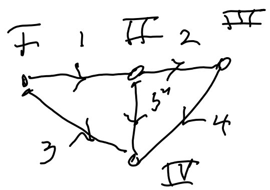
ENEE 610
Tu 09/03/19

Note Title 9/2/2019

Definition of Laplacian for a graph $L = D - A_{dj}$

D = degree, A_{dj} = adjacency



$D = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$ $\quad A_{Dj} = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$ } nodes vs nodes

$L = \begin{bmatrix} 2 & -1 & 0 & -1 \\ -1 & 3 & -1 & -1 \\ 0 & -1 & 2 & -1 \\ -1 & -1 & -1 & 3 \end{bmatrix}$ singular

delete last row and column

$\det \begin{bmatrix} 2 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 2 \end{bmatrix} = 12 + 0 + 0 - 0 - 2 - 2$
$= 12 - 4 = 8 = \#$ of trees

A_a = incidence (augmented) $= \begin{array}{c|c} & \begin{matrix} 1 & 2 & 3 & 4 & 5 \end{matrix} \\ \begin{matrix} \text{I} \\ \text{II} \\ \text{III} \\ \text{IV} \end{matrix} & \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & -1 & -1 \end{bmatrix} \end{array} \Rightarrow A = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 & 0 \end{bmatrix}$

$$AA^T = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 2 \end{bmatrix} = L$$

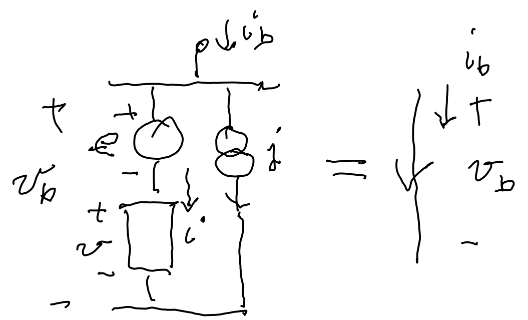
delete a row

$n =$ # nodes

$t =$ # tree branches $= n-1$ (for 1 separate part)

$l =$ # links

$b =$ # branches $= t + l$



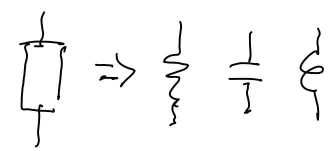
$$i_b = i + j = \mathcal{J}^{+} i_l$$

$$v_b = v + e = \mathcal{C}^T v_t$$

$$\Downarrow$$

$$i = \mathcal{J}^T i_l - j$$

$$v = \mathcal{C}^T v_t - e$$



$$Av = Bi$$

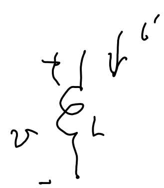
$$\underset{\sim}{0}_t = \begin{bmatrix} 1_t & K \end{bmatrix} i_b = \begin{bmatrix} 1_t & K \end{bmatrix} \begin{bmatrix} i_t \\ i_l \end{bmatrix} = i_t + K i_l \Rightarrow -i_t = K i_l$$

$$\begin{bmatrix} i_t \\ i_l \end{bmatrix} = \begin{bmatrix} -K \\ 1_l \end{bmatrix} i_l$$

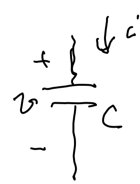
note $\mathcal{J} = \begin{bmatrix} -K^T & \vdots & 1_l \end{bmatrix}$

$$= \begin{bmatrix} -K^T & \vdots & 1_l \end{bmatrix}^T i_l$$

$$= \mathcal{J}^T i_l \quad \Longrightarrow i_b = \mathcal{J}^T i_l \,, \quad v_b = \mathcal{C}^T v_t \text{ by duality}$$

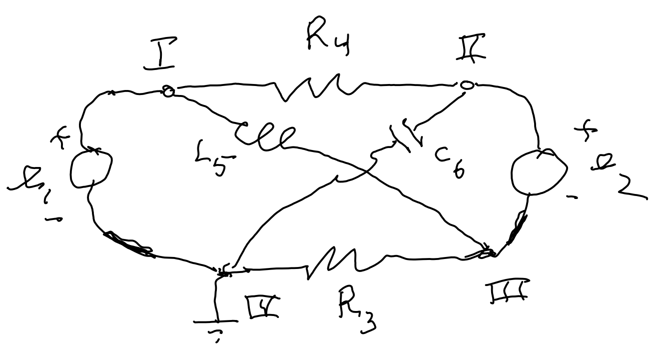


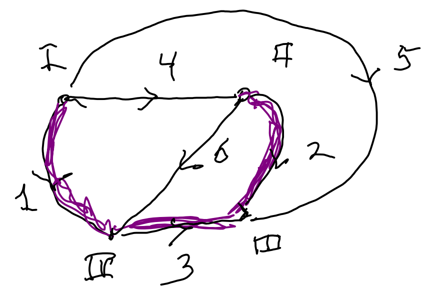
$$v = L \frac{di}{dt} = L\,\Delta \cdot i$$



$$i = \frac{dq}{dt} = \frac{d\,Cv}{dt}$$
$$= C \frac{dv}{dt} = C\,\Delta\, v$$

Ex:





— Tree branch

$G = 1/R$

$$e = \begin{bmatrix} e_1 \\ e_2 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad j = \underline{0}_6$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & G_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & G_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \alpha C_6 \end{bmatrix} v = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \alpha L_5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} i \quad \Longrightarrow A(\alpha)\, v = B(\alpha)\, i$$

$$\text{KCL} \Rightarrow \begin{array}{c} 1 \\ 2 \\ 3 \end{array}\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & -1 \end{bmatrix} i_b, \quad \begin{array}{c} 4 \\ 5 \\ 6 \end{array}\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 & 1 & -1 & 1 & 0 & 0 \\ -1 & 0 & -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 \end{bmatrix} v_b$$

K $\qquad -K^T$

$$v = v_b - e = C^T v_t - e \quad , \quad i = i_b - j = \mathcal{J}^T i_\ell - j$$

$$A(C^T v_t - e) = B(\mathcal{J}^T i_\ell - j) \Rightarrow \underbrace{AC^T}_{(b \times t)} v_t - \underbrace{B\mathcal{J}^T}_{(b \times \ell)} i_\ell = Ae - Bj$$

$$\underbrace{\begin{bmatrix} \underbrace{AC^T}_{b \times t} & -B\mathcal{J}^T \end{bmatrix}}_{b \times b} \underbrace{\begin{bmatrix} v_t \\ i_\ell \end{bmatrix}}_{\substack{b \times 1 \\ \text{unknowns}}} = \underbrace{Ae - Bj}_{b \times 1 \text{ vectors}}$$

$$
\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & G_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & G_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \ell C_6 \end{bmatrix}
\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix}
=
\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & G_3 \\ G_4 & -G_4 & G_4 \\ 1 & 0 & 1 \\ 0 & \ell C_6 & -\ell C_6 \end{bmatrix}
= A\ell^T \quad ; \quad A\ell = \begin{bmatrix} \ell_1 \\ \ell_2 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
$$

$$
\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \ell L_5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}
\begin{bmatrix} -1 & -1 & 0 \\ 1 & 0 & -1 \\ -1 & -1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
=
\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & -1 & 1 \\ 1 & 0 & 0 \\ 0 & \ell L_5 & 0 \\ 0 & 0 & 1 \end{bmatrix}
= Bq^T
$$

circuit equations

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & G_3 & +1 & +1 & -1 \\ G_4 & -G_4 & G_4 & -1 & 0 & 0 \\ 1 & 0 & 1 & 0 & -sL_5 & 0 \\ 0 & sC_6 & -sC_6 & 0 & 0 & -1 \end{array}\right] \begin{bmatrix} v_{1t} \\ v_{2t} \\ v_{3t} \\ i_{4l} \\ i_{5l} \\ i_{6l} \end{bmatrix} = \begin{bmatrix} e_1 \\ e_2 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$s \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -L_5 & 0 \\ 0 & C_6 & -C_6 & 0 & 0 & 0 \end{bmatrix} x = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -G_3 & -1 & -1 & +1 \\ -G_4 & G_4 & -G_4 & 1 & 0 & 0 \\ -1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} x + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}$$

$$sEx = Ax + Bu, \quad u = \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \text{input}$$