Note Title 10/25/2016

EE 610
10/25/16

Butterworth filters, p. 397 $\equiv$ maximally flat low-pass

$$T(s) = \frac{K}{s^m + a_{m-1}s^{m-1} + \dots + a_1 s + a_0} = \frac{K}{D(s)} \qquad D(s) = \text{Hurwitz}$$

$\dfrac{d^k|T(j\omega)|}{d\omega^k}$ has all powers possible $= 0$

$$\frac{d|T(j\omega)|^2}{d\omega} = 2|T(\omega)| \cdot \frac{d|T(j\omega)|}{d\omega} = 0$$

$$\Rightarrow \frac{d|T(\omega)|}{d\omega} = 0$$

$$\frac{d\, 1/|T(j\omega)|^2}{d\omega} = -\frac{1}{|T(j\omega)|^4} \cdot 2|T(j\omega)| \cdot \frac{d|T(j\omega)|}{d\omega} = 0 \Rightarrow \frac{d|T(j\omega)|}{d\omega} = 0$$

$$\frac{1}{|T(j\omega)|^2} = \frac{1}{T(j\omega)} \cdot \frac{1}{T^*(j\omega)} = |(j\omega)^m + \dots + a_1 j\omega + a_0|^2$$

$$= (j^m\omega^m + a_{m-1}(j\omega)^{m-1} + a_1 j\omega + a_0)((-j\omega)^m + a_{m-1}(-j\omega)^{m-1} - a_1 j\omega + a_0)$$

$$\frac{d}{d\omega}\frac{1}{|T(j\omega)|^2} = \left(m(j)^m \omega^{m-1} + \cdots + a_1 j\right)\left((-j)^m \omega^m + \cdots - a_1 j\omega + a_0\right)$$

$$\left(j^m \omega^m + \cdots + a_1 j\omega + a_0\right)\left((-j)^m \omega^{m-1} + \cdots - a_1 j\right)$$

$$= 2m\omega^{2m-1} + \cdots + 2a_1^2\omega = \omega\left(2m\omega^{2m-2} + 2a_1^2\right)\omega \Rightarrow 0$$

$$\frac{1}{|T(j\omega)|^2} = b_0 + b_2\omega^2 + \cdots + b_{2m}\omega^{2m}$$ setting all possible $\Rightarrow$

$$= b_0 + b_{2m}\omega^{2m} \qquad = b_2, \ldots, b_{2m-2} = 0$$

$b_0 \;\&\; b_{2m} \geq 0$ desire zeros of this

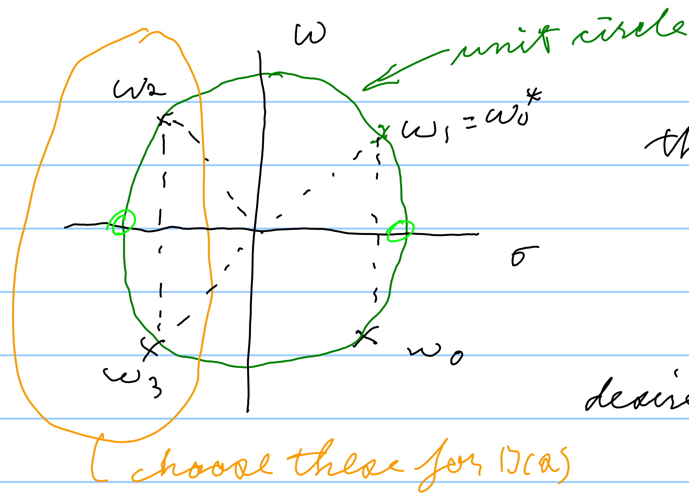
$$b_{2m}\omega^{2m} + b_0 = 0 \Rightarrow \omega^{2m} = -b_0/b_{2m}$$ normalize to $b_0 = 1 = b_{2m}$

$$\omega_k^{2m} = -1 = +e^{j\pi} = e^{j\pi + j2k\pi}, \quad k = 0, 1, \ldots$$

$$\omega_k = e^{j\left(\frac{\pi(2k-1)}{2m}\right)}$$ $|\omega_k| = 1 \Leftarrow$ these roots are on the unit circle

$m = 2$; $k = 0$, $\omega_0 = e^{j(-\pi/4)}$, $k = 1$, $\omega_1 = e^{j\pi/4} = \omega_0^*$

$k=2, \quad \omega_2 = e^{j3\pi/4}$, $\quad k=3, \quad \omega_3 = e^{j5\pi/4} \equiv \omega_2^*, \quad \omega_4 = e^{j7\pi/4} \equiv e^{j(-\pi/4)}$



these are for $T(s)\,T(-s) = \dfrac{K^2}{(s-\omega_0)(s-\omega_1)(s-\omega_2)(s-\omega_3)}$

$$|T(j\omega)|^2 = \frac{K}{1+\omega^{2m}}$$

desire $T(s) = \dfrac{K}{D(s)} \Leftarrow$ Hurwitz

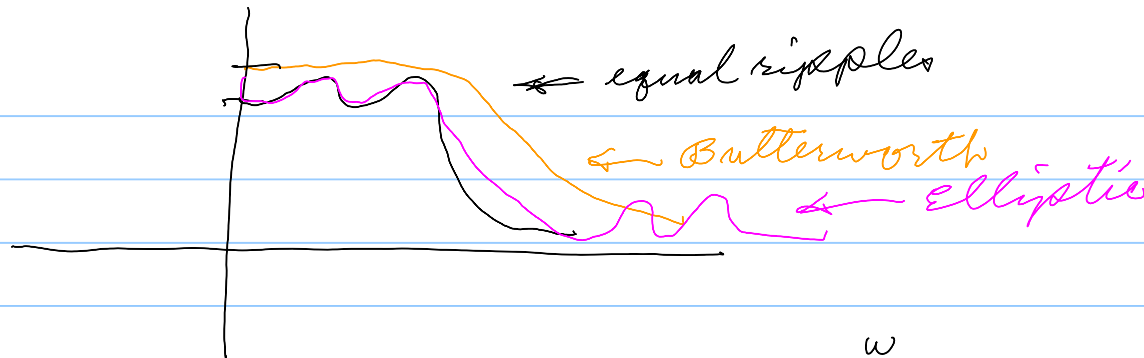
$$D(s) = (s-\omega_2)(s-\omega_3) = s^2 - (\omega_2+\omega_3)s + \omega_2\omega_3,$$

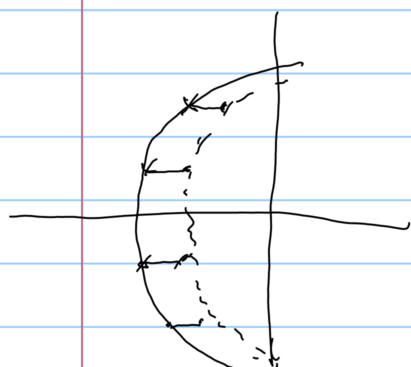
$\omega_3 = \omega_2^*$

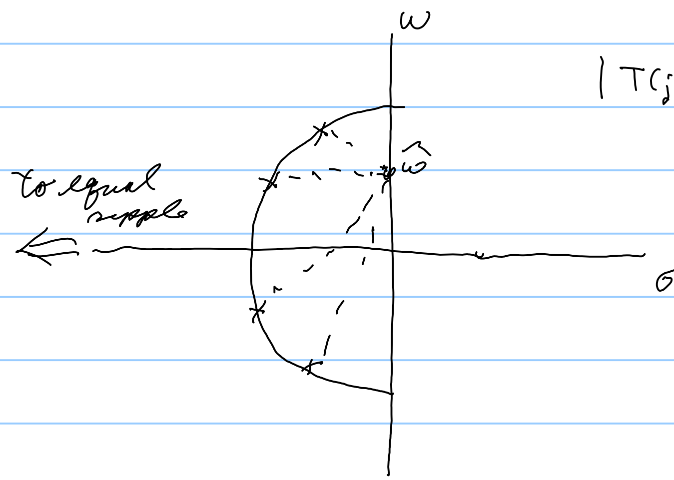
$$= s^2 - (2\cos(3\pi/4)s) + |\omega_2|^2$$

$$D(s) = s^2 + 2/\sqrt{2}\, s + 1 = s^2 + \sqrt{2}\, s + 1 \Rightarrow \text{Butterworth polynomial}$$

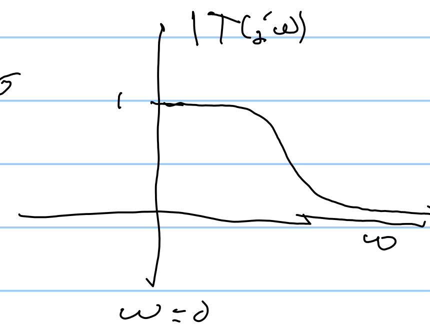
Equal ripple filters Chebyshev polynomials



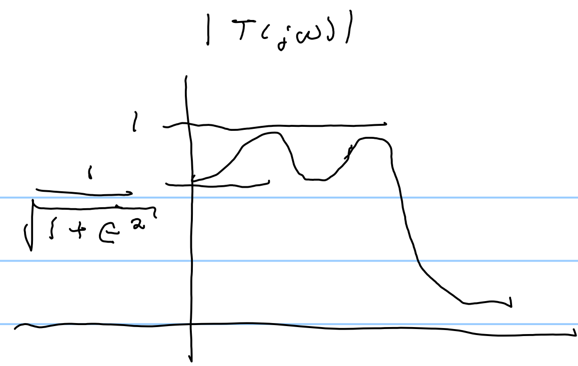




$$|T(j\omega_0)| = \frac{1}{|j\hat{\omega}-\omega_2||j\hat{\omega}-\omega_3|}$$



distance to move is $\tanh v$, $v = \frac{1}{n} \sinh^{-1} \frac{1}{\epsilon}$



$|T(j\omega)|$
1
$\frac{1}{\sqrt{1+\epsilon^2}}$