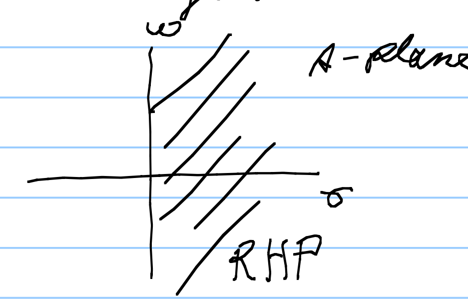
EE 610

10/09/14

Note Title 10/6/2014

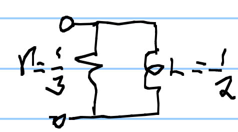
Positive Real functions; PR = rational positive real = physically realizable



Definition of Positive Real (of $s = \sigma + j\omega$); $y(s)$

1. $y(s)$ is real for $\sigma \geq 0$ (real circuits)
2. $y(s)$ is analytic in $\sigma > 0$ (stable circuits)
3. $\operatorname{Re} y(s) \geq 0$ in $\sigma > 0$ (passive)

Ex: $y(s) = \dfrac{3s+2}{s} = 3 + \dfrac{2}{s}$



is passive

$$y(s) = 3 + \frac{2}{\sigma + j\omega} = 3 + \frac{2(\sigma - j\omega)}{(\sigma^2 + \omega^2)} \Rightarrow \operatorname{Re} y(s) = 3 + \frac{2\sigma}{\sigma^2 + \omega^2} > 0 \text{ in } \sigma > 0$$

Ex $\quad y(s) = e^{s} \qquad\qquad y(\sigma) = e^{\sigma}$ is real

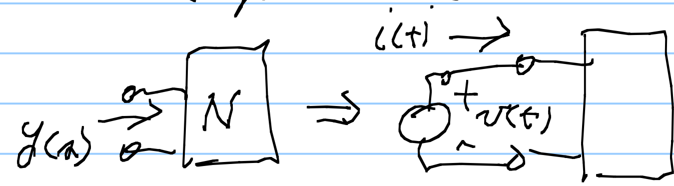
$y(s)$ is entire ∴ is analytic in $\sigma > 0$

$= e^{\sigma} \cdot e^{j\omega}$

$\mathrm{Re}\, y(s) = e^{\sigma}\left[\dfrac{e^{j\omega} + e^{-j\omega}}{2}\right] = e^{\sigma}\cos\omega \quad$ can be < 0, say $\omega = \pi$

∴ e^{s} is not positive real

Passivity



$\mathrm{Power}_{in}(t) = v(t)\, i(t)$

$2\,\mathrm{Power}_{in}(t) = v^{*}(t)\, i(t) + v(t)\, i^{*}(t) = p(t)$

$\mathcal{E}(t)$ = energy into N $= \displaystyle\int_{-\infty}^{t} p(\tau)\, d\tau \geq 0$ for all t if passive

(definition of passive

$$\mathcal{E}(\infty) = \int_{-\infty}^{\infty} p(\tau)\, d\tau \geq 0$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} \left[v^*(\tau)\, i(\tau) + v(\tau)\, i^*(\tau) \right] d\tau$$

Parseval's theorem

$$= \frac{1}{2}\int_{-\infty}^{\infty} \left[V^*(j2\pi f)\, I(j2\pi f) + V(j2\pi f)\, I^*(j2\pi f) \right] df$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} \left[V^* y(j\omega) V + V\, y^*(j\omega)\, V^* \right] \frac{d\omega}{2\pi}$$

$$= \int_{-\infty}^{\infty} V^* \left[\frac{y(j\omega) + y^*(j\omega)}{2} \right]^T V \frac{d\omega}{2\pi} > 0$$

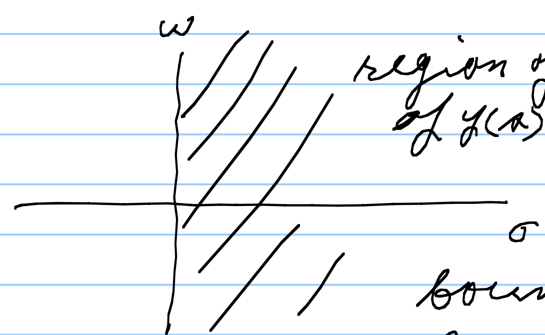
T = transpose

$$= \int_{-\infty}^{\infty} |V|^2 \operatorname{Re} y(j\omega) \frac{d\omega}{2\pi} > 0 \quad \text{for all } V$$

$\Rightarrow \operatorname{Re} y(j\omega) \geq 0$ for all real ω

but need to observe $\operatorname{Re} y(s)$

look @ $e^{-y(s)}$ in $\sigma > 0$ apply maximum modulus theorem $\Rightarrow$ max. of the magnitude of a function $f(s)$ analytic in a region is on the boundary of the region.



region of analyticity of $y(s)$ & $e^{-y(s)}$

boundary is $s = j\omega$

$$\left|e^{-y(s)}\right| = \left|e^{-\operatorname{Re} y(s) - j \operatorname{Im} y(s)}\right| = \left|e^{-\operatorname{Re} y(s)}\right| \leq 1$$

$\Rightarrow \operatorname{Re} y(s) \geq 0$ in $\sigma > 0$ as the boundary has $\operatorname{Re} y(j\omega) > 0$

Given a PR $y(s)$ & we desire to form a circuit

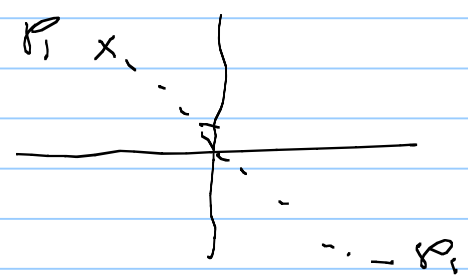
1st case: Lossless PR $\Rightarrow$ $\mathcal{E}(\infty) = 0$ $\Rightarrow$ $\mathrm{Re}\, y(j\omega) = 0$

$2\,\mathrm{Re}\, y(j\omega) = y(j\omega) + y(-j\omega) = 0$ let $\omega = s/j$ to analytically continue $y(j\omega)$ to $y(s)$

$\Rightarrow$ $y(s) + y(-s) \equiv 0$ for all s

(as zero on a dense set of ω's)

$y(s) = -y(-s)$ $\Rightarrow$ $\mathrm{Ev}\, y(s) \equiv 0$ $\Rightarrow$ allows synthesis by Richards' function



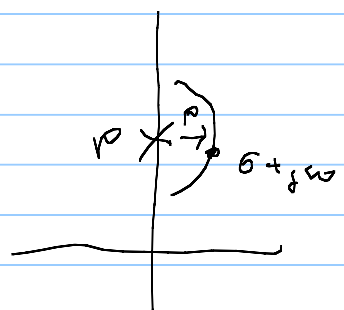
p_1 = pole of $y(s)$ then $-p_1$ is a pole of $y(-s)$

not allowed if $\sigma_1 \neq 0$ for PR

$\therefore$ all poles of a lossless PR function are on the $j\omega$ axis

(also $\frac{1}{y(s)} = z(s)$ is PR if $y(s)$ is PR) $\Rightarrow$ $y(s) + y^*(s) = \frac{1}{z} + \frac{1}{z^*} = \frac{z+z^*}{|z|^2}$

$\therefore$ all zeroes of a PR lossless $y(s)$ are on the $j\omega$ axis



$$y(s) = \frac{k}{(s-p)^n} + \Rightarrow \operatorname{Re} y(s)\Big|_{\text{near } p} = \rho^n \left(\cos\Big|_{n\left(-\frac{\pi}{2}\right)}^{\left(\frac{\pi}{2}\right)\times n} \right) + \ldots$$

$$\approx \rho^n e^{j\angle(s-p)^n}$$

$\downarrow$

$\cos \lessapprox 0$ if $n > 1$

if the pole is simple $y(s) \approx \frac{k}{(s-j\omega_1)}$

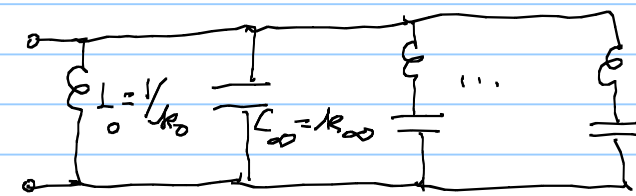
$$y(s) = \frac{k(\sigma - j(\omega-\omega_1))}{(\sigma+j\omega-j\omega_1)(\sigma-j\omega+j\omega_1)} \approx \frac{k\sigma}{|\rho|^2} + j\,\mathrm{Im}$$

$\mathrm{Re}\, y(s)\Big|_{\text{near pole in } \sigma>0} \approx \dfrac{k\sigma}{|\rho|^2} \geq 0$ if PR $\Longrightarrow$ $k > 0$ (real)

lossless

$$y(s) = \frac{k_1}{s-p_1} + \frac{k_1^*}{s-p_1^*} + \cdots + \frac{k_0}{s} + k_\infty s$$

$$= \frac{k_0}{s} + k_\infty s + \sum_{\ell=1}^{N} \frac{2k_\ell s}{s^2+\omega_\ell^2} \Longrightarrow$$



$$y_\ell(s) = \frac{2k_\ell s}{s^2+\omega_\ell^2} = \frac{1}{z_\ell(s)} = \frac{1}{\dfrac{s}{2k_\ell} + \dfrac{\omega_\ell^2}{2k_\ell s}}$$

$L_\ell = \dfrac{1}{2k_\ell}$, $C_\ell = \dfrac{2k_\ell}{\omega_\ell^2}$

here $\dfrac{1}{L_\ell C_\ell} = \omega_\ell^2$

2nd Foster design