EE 610
10/08/13

Note Title 10/4/2013

pr functions: If $y(s)$ is pr then $\frac{1}{y(s)} = z(s)$ is also pr

If $y_1(s)$ & $y_2(s)$ are pr then $y_1(s) + y_2(s)$ is pr

& $y_1(y_2(s))$ is pr

but $y_1(s) \times y_2(s)$ need not be pr

Ex: $c_1 s \times c_2 s = c_1 c_2 s^2$ & s^2 is no pr

Conditions (definition)

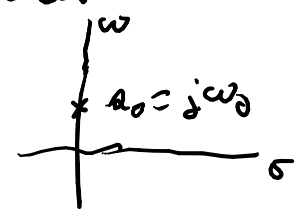
1) $y(s)$ is real for real s, for $\operatorname{Re} s > 0$.
(if rational $\Rightarrow$ real coefficients)

2) $y(s)$ is analytic in $\operatorname{Re} s > 0$
(if rational no poles in RHP of s)

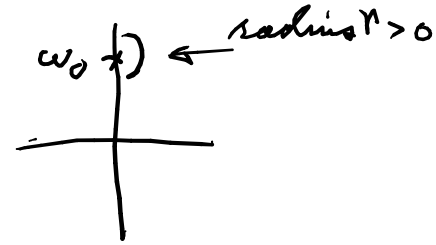
3) $\operatorname{Re} y(s) \geq 0$ in $\operatorname{Re} s > 0$

Properties:
Poles on $j\omega$ axis are simple with real positive residues

Reason: $y(s) \approx \frac{k}{(s-s_0)^n} + \cdots$ near pole



$\operatorname{Re} y(s)$, $\sigma > 0$, $s = \sigma + j\omega$



$$y(s) \approx \frac{|k| e^{j\angle k}}{(\sigma + j\omega - j\omega_0)^n}$$

$$\approx \frac{|k| e^{j\angle k}}{r^n e^{j\angle \text{ on circle} \times n}}, \quad -\frac{\pi}{2} < \angle s - j\omega_0 < \pi/2$$

$$\operatorname{Re} y(s) = \frac{|k|}{r^n} \cos(\angle k - n\angle(s - j\omega_0))$$

need this ≥ 0 this angle, can vary only between $-\pi/2$ & $+\pi/2$

$\Rightarrow n = 1,\ \Delta k = 0 \Rightarrow$ simple pole &
$k > 0$ real

$\Rightarrow$ zeros on $j\omega$ axis are simple.

Lossless! (1 port) (passive) $\quad P_{ave}(j\omega) = 0 = \mathrm{Re}(V^* I) = \underline{V^* y(j\omega) V + (V^{**} y^*(j\omega) V^*)}$

$$= \frac{1}{2} V^*[y(j\omega) + y^*(j\omega)] V^2$$

$$\Rightarrow \frac{1}{2}[y(j\omega) + y^*(j\omega)] = 0$$

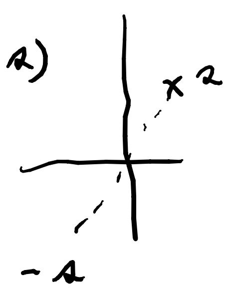
$$= \frac{1}{2}[y(j\omega) + y(-j\omega)]$$ if pr as $f^*(j\omega) = f((j\omega)^*)$

$\Rightarrow$ let $\omega \to s/j \Rightarrow \frac{1}{2}[y(s) + y(-s)]$ this is analytic in s (except at possibly poles) so in a dense set on $s = j\omega$ (if rational)

by analytic continuation

$\mathrm{Ev}(y(s)) \equiv 0$ if $y(s)$ is p.r. & lossless

$\Rightarrow y(s) = -y(-s)$



(rules out poles in $\sigma > 0$)

given $f(x)$ form

$$\left.\begin{aligned} 2\,\mathrm{Ev}(f(x)) &= f(x) + f(-x) \\ 2\,\mathrm{Od}(f(x)) &= f(x) - f(-x) \end{aligned}\right\} \quad f(x) = \mathrm{Ev}(f(x)) + \mathrm{Od}(f(x))$$

Given a lossless pr $y(s)$ then all poles are on the $j\omega$ axis

$$y(s) = \frac{k_1}{s - j\omega_1} + \frac{k_1}{s + j\omega_1} + \cdots + \frac{k_n}{(s - j\omega_n)} + \frac{k_n}{(s + j\omega_n)} + \frac{k_0}{s} + k_\infty s$$

$$= \frac{2k_1 s}{s^2 + \omega_1^2} + \cdots + \frac{2k_n s}{s^2 + \omega_n^2} + \frac{k_0}{s} + k_\infty s$$

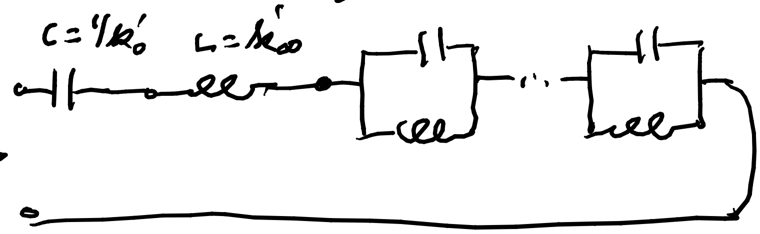
1st Foster $z(s) = \frac{k_0'}{s} + k_\infty' s + \sum_{i=1}^{n'} \frac{2k_i' s}{s^2 + \omega_i'^2}$

(these poles are zeros of $y(s) = 1/z(s)$)

$= y(s) \rightarrow$

$\parallel$

$1/z(s)$

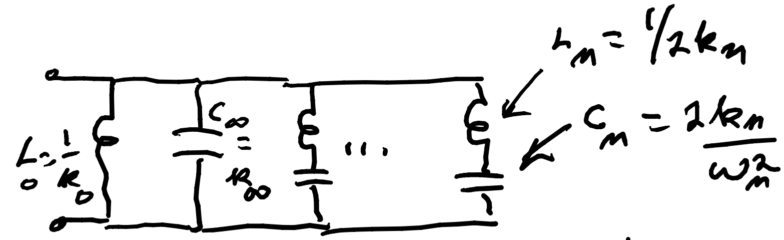


1st Foster = partial fraction expansion of $z(s)$

$$y(s) = \frac{k_0}{s} + k_\infty s + \sum_{i=}^{n} \frac{2k_i s}{s^2 + \omega_i^2}$$

$y(s) \rightarrow$

(lossless PR)



$$y_\ell(s) = \frac{2k_\ell s}{s^2 + \omega_\ell^2} = \frac{1}{\frac{s}{2k_\ell} + \frac{\omega_\ell^2}{2k_\ell s}}$$

(2nd Foster form)

Poles

$$y(j\omega) = \frac{k_0}{j\omega} + jk_\infty \omega + \sum_{i=1}^{m} \frac{2k_i(j\omega)}{-\omega^2 + \omega_i^2}$$

$$= j\,B(\omega) = j\left[-\frac{k_0}{\omega} + k_\infty \omega + \sum \frac{2k_i\omega}{-\omega^2+\omega_i^2}\right]$$

$$\frac{dB(\omega)}{d\omega} = \frac{k_0}{\omega^2} + k_\infty + \sum \underbrace{\left(\frac{2k_i}{-\omega^2+\omega_i^2} - \frac{2k_i\omega(-2\omega)}{(-\omega^2+\omega_i^2)^2}\right)}_{\frac{2k_i}{(-\omega^2+\omega_i^2)^2}\left[\underbrace{(-\omega^2+\omega_i^2)+2\omega^2}_{\omega^2+\omega_i^2}\right]}$$

≥ 0 for $-\infty < \omega < \infty$