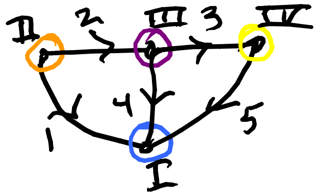
EE 610
09/12/13

Incidence matrix



$$\begin{bmatrix}0\\0\\0\\0\end{bmatrix} = \underbrace{\begin{bmatrix}1&0&0&1&1\\-1&-1&0&0&0\\0&1&-1&-1&0\\0&0&1&0&-1\end{bmatrix}}\begin{bmatrix}i_1\\i_2\\i_3\\i_4\\i_5\end{bmatrix}$$

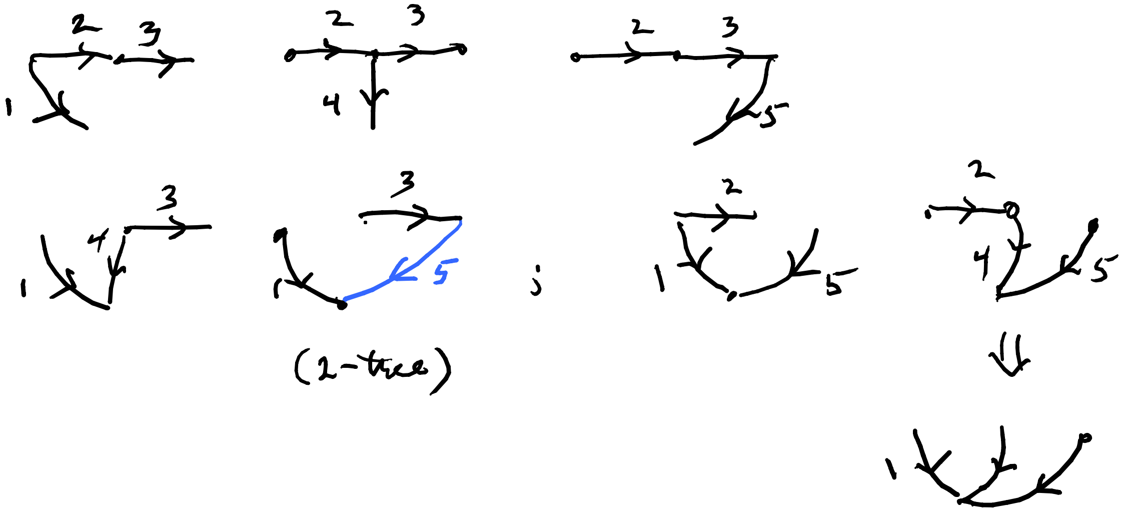
$= A =$ incidence matrix $= n \times b$ singular matrix

delete a row

$$A_a = \begin{bmatrix}1&0&0&1&1\\-1&-1&0&0&0\\0&1&-1&-1&0\end{bmatrix}$$

$$A_a A_a^T = \begin{bmatrix}1&0&0&1&1\\-1&-1&0&0&0\\0&1&-1&-1&0\end{bmatrix}\begin{bmatrix}1&-1&0\\0&-1&1\\0&0&-1\\1&0&-1\\1&0&0\end{bmatrix} = \begin{bmatrix}3&-1&-1\\-1&2&-1\\-1&-1&3\end{bmatrix}$$

$\det(A_a A_a^T) = 18-1-1-2-3-3 = 8$ = # of trees in our graph



Equations of a circuit

KVL, KCL: $v_b = e^T v_t$, $0 = e\, i_b$,

$i_b = \mathfrak{J}^T i_\ell$, $0 = \mathfrak{J} v_b$

$v_b = v + e$
$i_b = i + j$; if linear $Av = Bi$

A & B are assumed to be $b \times b$

$$v = v_b - e \Rightarrow Av = Av_b - Ae$$

$$i = i_b - j \Rightarrow Bi = Bi_b - Bj$$

$$Av_b - Ae = Bi_b - Bj \Rightarrow Av_b - Bi_b = Ae - Bj$$

$$\left[Ae^T v_t - BJ^T i_\ell\right] = A\ell - Bj$$

$$\left[Ae^T \quad -BJ^T\right]\begin{bmatrix} v_t \\ i_\ell \end{bmatrix} = A\ell - Bj \qquad ; \; x = \text{unknowns} = \begin{bmatrix} v_t \\ i_\ell \end{bmatrix}$$

$$\left[Ae^T \quad -BJ^T\right] x = A\ell - Bj$$

To solve, invert $\left[Ae^T \quad -BJ^T\right]$ this is $b \times (t+\ell) = b \times b$

i.e. b eqs in b unknowns

(can solve if $\left[Ae^T \quad -BJ^T\right]$ is non-singular)

For our example:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & Cs & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{bmatrix} = \begin{bmatrix} r_d & 0 & 0 & 0 & 0 \\ 0 & R_1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & R_2 & 0 \\ 0 & 0 & 0 & 0 & sL \end{bmatrix}\begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \\ i_5 \end{bmatrix}$$

$$e = \left[\begin{array}{ccc|cc} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & -1 & -1 \\ \underbrace{0 \quad 0 \quad 1}_{1_3} & & & \underbrace{0 \quad -1}_{K} & \end{array}\right], \quad J = \begin{bmatrix} -1 & 1 & 0 & 1 & 0 \\ -1 & 1 & 1 & 0 & 1 \end{bmatrix}$$

$$Ae^T = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & Cs & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \\ 1 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & Cs \\ 1 & -1 & 0 \\ 1 & -1 & -1 \end{bmatrix}$$

$$B\mathcal{Q}^T = \begin{bmatrix} r_d & 0 & 0 & 0 & 0 \\ 0 & R_1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & R_2 & 0 \\ 0 & 0 & 0 & 0 & sL \end{bmatrix} \begin{bmatrix} -1 & -1 \\ 1 & 1 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -r_d & -r_d \\ R_1 & R_1 \\ 0 & 1 \\ R_2 & 0 \\ 0 & sL \end{bmatrix}, \quad As = A\begin{bmatrix} E \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}; 5\times 1 = \begin{bmatrix} E \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

$$\mathcal{J} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$[As^T - B\mathcal{Q}^T] \Rightarrow \begin{bmatrix} 1 & 0 & 0 & +r_d & +r_d \\ 0 & 1 & 0 & -R_1 & -R_1 \\ 0 & 0 & Cs & 0 & -1 \\ 1 & -1 & 0 & -R_2 & 0 \\ 1 & -1 & -1 & 0 & -sL \end{bmatrix} x = \begin{bmatrix} E \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}; \quad x = \begin{bmatrix} v_{1_b} \\ v_{2_b} \\ v_{3_b} \\ i_{4_b} \\ i_{5_b} \end{bmatrix}$$

as $s = d/dt$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & C & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -L \end{bmatrix} \frac{d}{dt} x = \begin{bmatrix} -1 & 0 & 0 & -r_d & -r_d \\ 0 & -1 & 0 & +R_1 & +R_1 \\ 0 & 0 & 0 & 0 & 1 \\ -1 & 1 & 0 & R_2 & 0 \\ -1 & 1 & 1 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} [E]$$

and for outputs y, $y = Cx$

$$\Rightarrow \boxed{\begin{aligned} E_x \frac{d}{dt} x &= A_x x + B_x u \\ y &= Cx \end{aligned}}, \quad u = [E]$$

known as
differential-algebraic
semi-state
singular
(state-variable if $E_x = 1_b$)

can eliminate non-derivative rows sometimes