Homework #6 ENEE 610

Problem #1

$$y_1(s) = \frac{s(s^2+4)}{(s^2+1)(s^2+5)} = \frac{s^3+4s}{s^4+6s^2+5} \qquad y_2(s) = \frac{(s^2+1)(s^2+5)}{s(s^2+4)} = \frac{s^4+6s+5}{s^3+4s}$$

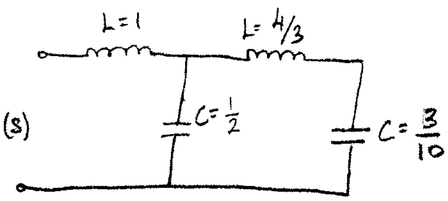
$y_1(s)$

1st Cauer

$$Z_1(s) = \frac{s^4+6s^2+5}{s^3+4s}$$

$$\begin{array}{l}
s^3+4s \,\overline{)\, s^4+6s^2+5} \quad s \\
\quad s^4+4s^2 \\
\quad 2s^2+5 \,\overline{)\, s^3+4s} \quad \frac{s}{2} \\
\qquad s^3+\frac{5}{2}s \\
\qquad \frac{3}{2}s \,\overline{)\, 2s^2+5} \quad \frac{4}{3}s \\
\qquad\quad 2s^2 \\
\qquad\quad 5 \,\overline{)\, \frac{3}{2}s} \quad \frac{3}{10}s \\
\qquad\qquad \frac{3}{2}s \\
\qquad\qquad 0
\end{array}$$

$$Z_1(s) = s + \cfrac{1}{\frac{1}{2}s + \cfrac{1}{\frac{4}{3}s + \cfrac{1}{\frac{3}{10}s}}}$$

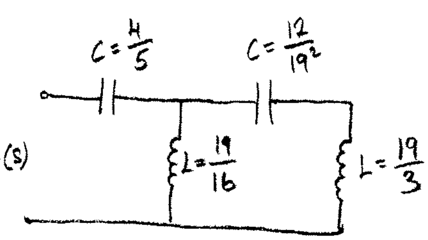


2nd Cauer

$$Z_1(s) = \frac{5+6s^2+s^4}{4s+s^3}$$

$$\begin{array}{l}
4s+s^3 \,\overline{)\, 5+6s^2+s^4} \quad \frac{5}{4s} \\
\quad 5+\frac{5}{4}s^2 \\
\quad \frac{19}{4}s^2+s^4 \,\overline{)\, 4s+s^3} \quad \frac{16}{19s} \\
\qquad 4s+\frac{16}{19}s^3 \\
\qquad \frac{3}{19}s^3 \,\overline{)\, \frac{19}{4}s^2+s^4} \quad \frac{19^2}{12s} \\
\qquad\quad \frac{19}{4}s^2 \\
\qquad\quad s^4 \,\overline{)\, \frac{3}{19}s^3} \quad \frac{3}{19s} \\
\qquad\qquad \frac{3}{19}s^3 \\
\qquad\qquad 0
\end{array}$$

$$Z_1(s) = \frac{5}{4s} + \cfrac{1}{\frac{16}{19s} + \cfrac{1}{\frac{19^2}{12s} + \cfrac{1}{\frac{3}{19s}}}}$$



1st Foster

$$Z_1(s) = \frac{(s^2+1)(s^2+5)}{s(s^2+4)} = k_\infty s + \frac{k_0}{s} + \frac{2k_2 s}{s^2+4}$$

$$(s^2+1)(s^2+5) = k_\infty s^2(s^2+4) + k_0(s^2+4) + 2k_2 s^2$$

$s=0$ $\quad 5 = 4k_0 \longrightarrow k_0 = \frac{5}{4}$

$s^2=-4$ $\quad -3 = -8k_2 \longrightarrow k_2 = \frac{3}{8}$

$s^2=1$ $\quad 12 = 5k_\infty + 5k_0 + 2k_2 \longrightarrow k_\infty = 1$

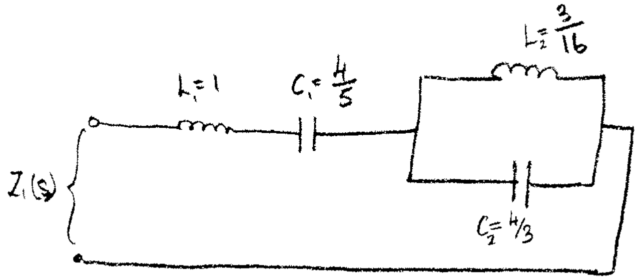
$\therefore$

$$Z_1(s) = s + \frac{5}{4s} + \frac{2\left(\frac{3}{8}\right)s}{s^2+4}$$

$L_1 = k_\infty \qquad L_2 = \frac{2k_2}{\omega_2^2} \qquad C_2 = \frac{1}{2k_2}$

$C_1 = \frac{1}{k_0}$

$\omega_2^2 = 4$



2nd Foster

$$y_1(s) = \frac{s(s^2+4)}{(s^2+1)(s^2+5)} = k'_\infty s + \frac{k'_0}{s} + \frac{2k'_2 s}{s^2+1} + \frac{2k'_4 s}{s^2+5}$$

$$s(s^2+4) = k'_\infty s(s^2+1)(s^2+5) + \frac{k'_0}{s}(s^2+1)(s^2+5) + 2k'_2 s(s^2+5) + 2k'_4 s(s^2+1)$$

$s^2=-1$ $\quad 3\sqrt{-1} = 2k'_2\sqrt{-1}\,(4) \longrightarrow k'_2 = \frac{3}{8}$

$s^2=-5$ $\quad -1\sqrt{-1} = 2k'_4\sqrt{-1}\,(-4) \longrightarrow k'_4 = \frac{1}{8}$

$s=1$ $\quad 5 = 12k'_\infty + 12k'_0 + 5 \longrightarrow k'_0 = -k'_\infty$

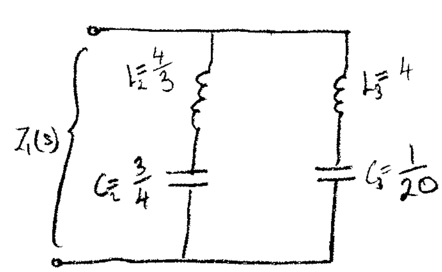
$s=2$ $\quad 16 = 90k'_\infty + \frac{45}{2}k'_0 + 16 \longrightarrow k_0 = -2k'_\infty$

$k'_0 = k'_\infty = 0$

$$\therefore \quad y_1(s) = \frac{2\left(\frac{3}{8}\right)s}{s^2+1} + \frac{2\left(\frac{1}{8}\right)s}{s^2+5}$$

$\omega_2^2 = 1 \qquad C_1 = k'_\infty = 0 \qquad L_2 = \frac{1}{2k'_2} \qquad L_3 = \frac{1}{2k'_4}$

$\omega_4^2 = 5 \qquad L_1 = \frac{1}{k'_0} = \infty \qquad C_2 = \frac{2k_2}{\omega_2^2} \qquad C_3 = \frac{2k_4}{\omega_4^2}$

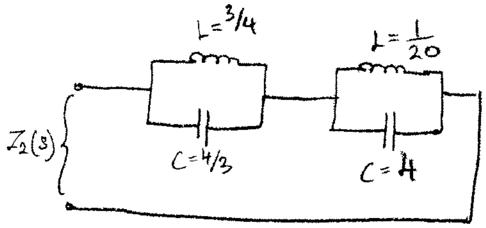


$y_2(s)$

1^{st} Foster

$Z_2(s) = y_1(s)$ $\therefore$ use result of 2^{nd} Foster on $y_1(s)$:

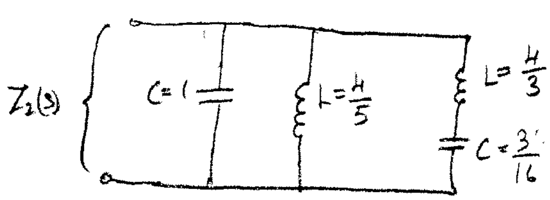
$$Z_2(s) = \frac{\frac{3}{4}s}{s^2+1} + \frac{\frac{1}{4}s}{s^2+5} = \frac{1}{\frac{4}{3}s + \frac{4}{3s}} + \frac{1}{4s + \frac{20}{s}}$$



2^{nd} Foster

$y_2(s) = Z_2(s)$ $\therefore$ use result of 1^{st} Foster on $y_1(s)$:

$$y_2(s) = s + \frac{5}{4s} + \frac{\frac{3}{4}s}{s^2+4} = s + \frac{5}{4s} + \frac{1}{\frac{4s}{3} + \frac{16}{3s}}$$



1st Cauer

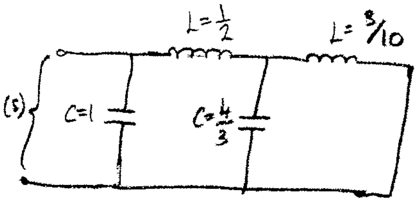
$$y_2(s) = \frac{s^4 + 6s^2 + 5}{s^3 + 4s}$$

(Use result of)

- perform continued fraction expansion exactly as 1st Cauer for $Z_1(s)$
- result is an admittance instead of Impedance:

$\therefore$

$$y_2(s) = s + \cfrac{1}{\frac{1}{2}s + \cfrac{1}{\frac{4}{3}s + \cfrac{1}{\frac{3}{10}s}}}$$



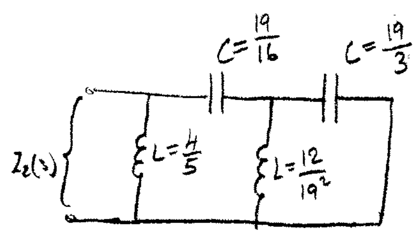
2nd Cauer

$$y_2(s) = \frac{5 + 6s^2 + s^4}{4s + s^3}$$

(Use result of)

- Perform continued fraction expansion exactly as 2nd Cauer for $Z_1(s)$:
- result is an admittance

$$\therefore\ y_2(s) = \frac{5}{4s} + \cfrac{1}{\frac{16}{19s} + \cfrac{1}{\frac{19^2}{12s} + \cfrac{1}{\frac{3}{19s}}}}$$



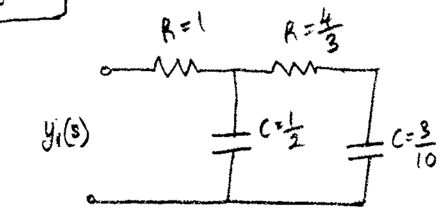
- Comparing the Cauer forms, we see that Capacitors and Inductors between y_1 and y_2 have been interchanged. Also, the leading component appears in parallel in y_2 while it would be in series in y_1.

- Comparing the Foster forms, we again see that the capacitor and inductor values have interchanged and series connections become parallel and vice versa. This is due to the fact that $y_1(s) = \frac{1}{y_2(s)}$

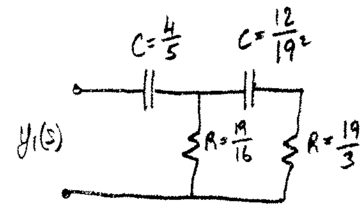
Problem #2

- Replacing each L by an R, we obtain the following circuits:

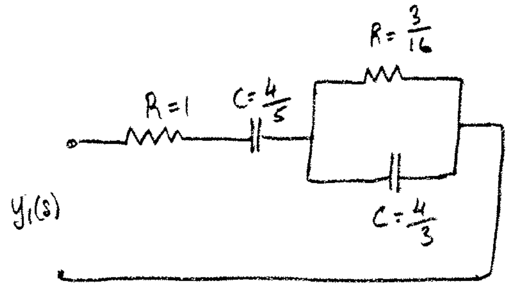
$\boxed{y_1(s)}$



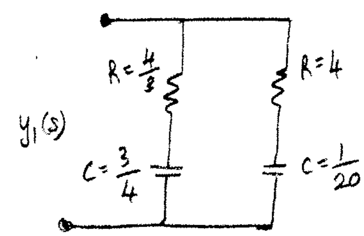
1st Cauer



2nd Cauer

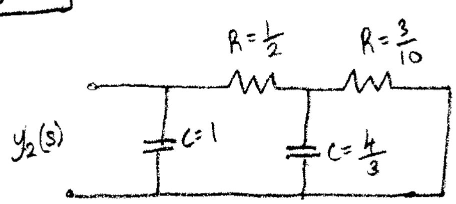


1st Foster

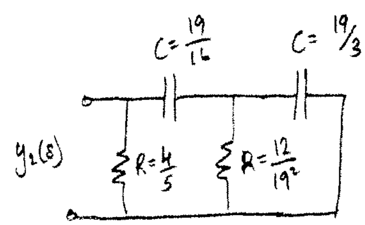


2nd Foster

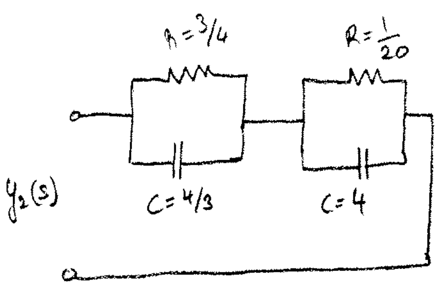
$\boxed{y_2(s)}$



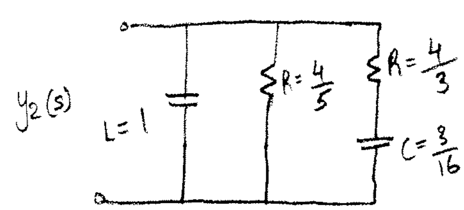
1st Cauer



2nd Cauer



1st Foster



2nd Foster

From each RC circuit we obtain:

$$y_1(s) = \frac{s(s+4)}{(s+1)(s+5)}$$

$$y_2(s) = \frac{(s+1)(s+5)}{s+4}$$

Compared to the original LC:

$$y_1(s) = \frac{s(s^2+4)}{(s^2+1)(s^2+5)}$$

$$y_2(s) = \frac{(s^2+1)(s^2+5)}{s(s^2+4)}$$

- for $y_1(s)$ the degree of the numerator and denominator have reduced compared to the original LC circuit
- the same thing is observed with $y_2(s)$, however we see that the denominator degree has dropped by two
- therefore $y_1(s)$ and $y_2(s)$ in the RC case are no longer inverses of each other.

Problem # 3

$$y(s) = \frac{s^2+s+2}{s^2+s+1} \qquad y(j\omega) = \frac{2-\omega^2+j\omega}{1-\omega^2+j\omega} = \frac{(2-\omega^2+j\omega)(1-\omega^2-j\omega)}{(1-\omega^2+j\omega)(1-\omega^2-j\omega)} = \frac{\omega^4-2\omega^2+2-j\omega}{\omega^4-\omega^2+1}$$

$$Re[y(j\omega)] = \frac{\omega^4-2\omega^2+2}{\omega^4-\omega^2+1} = f(\omega)$$

Find minimum:

$$\left.\frac{df}{d\omega}\right|_{\omega=\omega_0} = \frac{2\omega^3(\omega^2-2)}{(\omega^4-\omega^2+1)^2} = 0 \longrightarrow \omega_0 = \pm\sqrt{2},\ 0$$

from plot, $\omega=0$ is a maximum

$\therefore\ \omega_0 = \pm\sqrt{2}$

$$g = f(\omega_0) = \frac{2^2-4+2}{2^2-2+1} = \frac{2}{3}$$

Show $y(s)$ is PR

a) Find poles:

$s^2+s+1=0 \longrightarrow s_{1,2} = \frac{-1}{2} \pm j\frac{\sqrt{3}}{2}$

↳ no poles in RHP ✓

b) no poles on $j\omega$ axis ✓

c) $Re[y(j\omega)] = \frac{\omega^4-2\omega^2+2}{\omega^4-\omega^2+1} = \frac{(\omega^2-1)^2+1}{(\omega^2-\frac{1}{2})^2+\frac{3}{4}} \geq 0$ for $0 \leq \omega \leq \infty$ ✓

$\therefore\ y(s)$ is PR

Check zeroes/poles

$s^2+s+2 \longrightarrow s_{1,2} = -\frac{1}{2} \pm j\frac{\sqrt{7}}{2}$

$s^2+s+1 \longrightarrow s_{1,2} = -\frac{1}{2} \pm j\frac{\sqrt{3}}{2}$

no poles or zeroes on $j\omega$ axis

$Z_1(s)$

$$Z_1(s) = \frac{1}{y(s)-g} = \frac{1}{\frac{s^2+s+2}{s^2+s+1} - 2/3} = \frac{3(s^2+s+1)}{s^2+s+4}$$

Show $Z_1(s)$ is PR:

a) poles: $s_{1,2} = \frac{-1}{2} \pm j\frac{\sqrt{15}}{2}$

↳ no poles in RHP ✓

b) no poles on $j\omega$ axis ✓

c) $Re[Z_1(j\omega)] = \frac{3(\omega^2-2)^2}{\omega^4-7\omega^2+16} = \frac{3(\omega^2-2)^2}{(\omega^2-4)^2+\omega^2} \geq 0$ for $0 \leq \omega \leq \infty$ ✓

$\therefore\ Z_1(s)$ is PR

Even part zeroes

$$2\,\Sigma_v[y(s)] = y(s) + y(-s) = \frac{2(s^4+2s^2+2)}{s^4+s+1} = 0$$

$$\Rightarrow \quad s^4 + 2s^2 + 2 = 0$$

$$s_{1,2}^2 = -1 \pm j$$

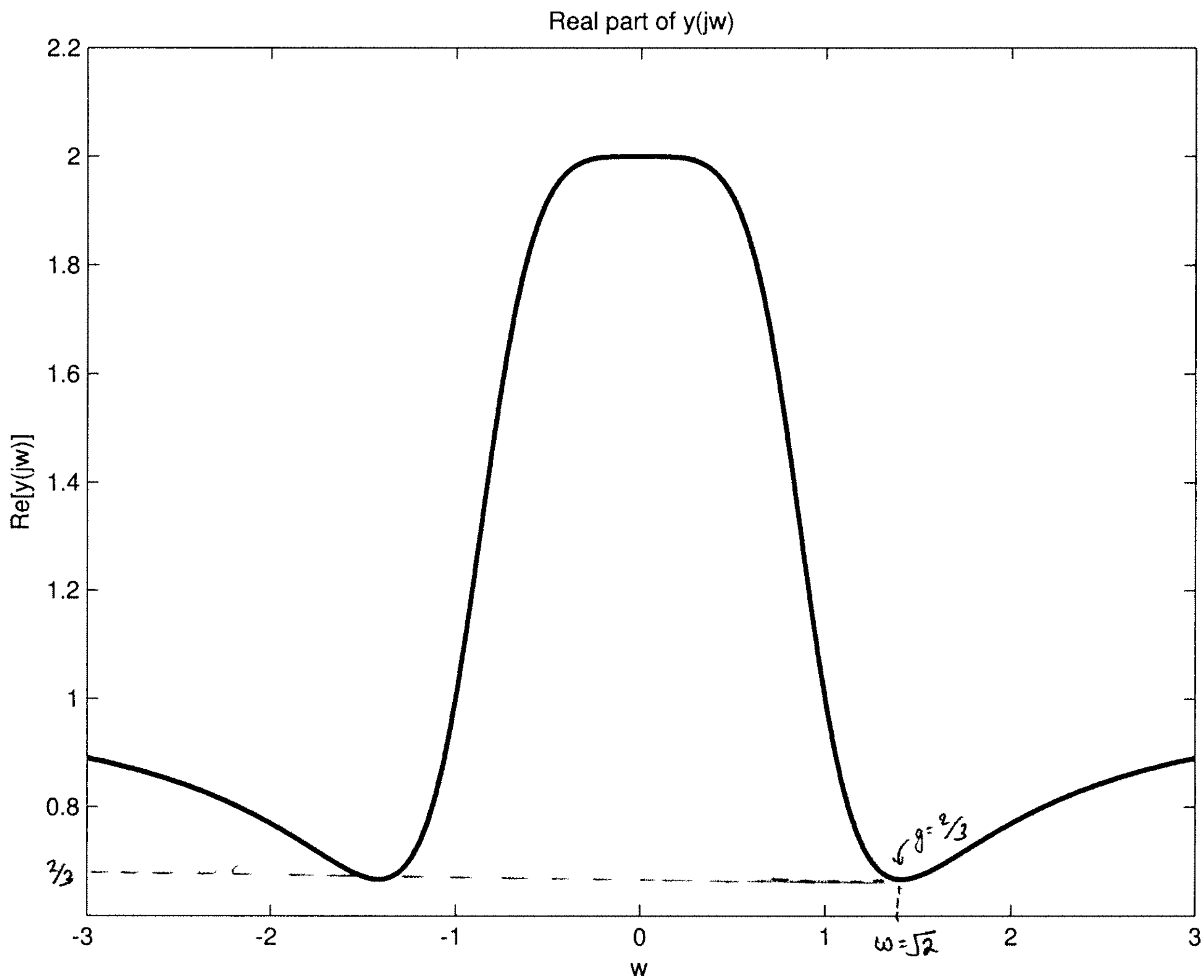
$$s_{1,2} = \pm\sqrt{-1 \pm j}$$

$$2\,\Sigma_v[z_1(s)] = z_1(s) + z_1(-s) = \frac{6(s^2+2)^2}{s^4+7s^2+16} = 0$$

$$(s^2+2)^2 = 0$$

$$s_{1,2}^2 = -2$$

$$s_{1,2} = \pm j\sqrt{2}$$



Real part of y(jw)
Re[y(jw)]
2.2
2
1.8
1.6
1.4
1.2
1
0.8
2/3
-3
-2
-1
0
1
2
3
w
g = 2/3
ω = √2

Problem # 4

$$y(s) = T(s) = \frac{3s^2+2s+1}{s^2+3} = 3 + \frac{2s-8}{s^2+3} = D + \frac{c_1 s + c_0}{s^2 + a_1 s + a_0}$$

- map to state varieble matrix form:

$$\begin{bmatrix}\dot{x}_1\\x_2\end{bmatrix} = \begin{bmatrix}0 & 1\\-a_0 & -a_1\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} + \begin{bmatrix}0\\1\end{bmatrix}u \Rightarrow \begin{bmatrix}\dot{x}_1\\x_2\end{bmatrix} = \begin{bmatrix}0 & 1\\-3 & 0\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} + \begin{bmatrix}0\\1\end{bmatrix}u \qquad (\dot{x} = Ax + Bu)$$

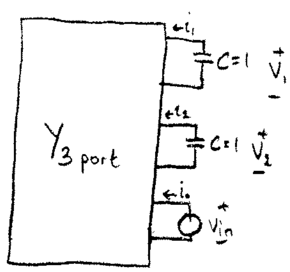
$$y = \begin{bmatrix}c_0 & c_1\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} + Du \Rightarrow y = \begin{bmatrix}-8 & 2\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} + 3u \qquad (y = Cx + Du)$$

- $T(s) = \frac{\text{output}}{\text{input}} = \frac{y}{u} \Rightarrow y = i_o \quad u = V_{in}$

- Set up 3-port admittance loaded in with unit capacitors:

$$\begin{bmatrix}-\dot{x}\\y\end{bmatrix} = \begin{bmatrix}-A & -B\\C & D\end{bmatrix}\begin{bmatrix}x\\u\end{bmatrix} \Rightarrow \begin{bmatrix}i_1\\i_2\\i_o\end{bmatrix} = \underbrace{\begin{bmatrix}0 & -1 & 0\\3 & 0 & -1\\-8 & 2 & 3\end{bmatrix}}_{Y_{3\,port}}\begin{bmatrix}V_1\\V_2\\V_{in}\end{bmatrix}$$

with corresponding circuit:



- Check result:

$$\left.\begin{aligned} i_1 &= -V_2\\ i_2 &= 3V_1 - V_{in}\\ i_o &= -8V_1 + 2V_2 + 3V_{in}\end{aligned}\right\} \text{from 3port admittance}$$

$$\left.\begin{aligned} i_1 &= -sCV_1 = -sV_1\\ i_2 &= -sCV_2 = -sV_2\end{aligned}\right\} \text{at ports 1 and 2}$$

$\therefore \quad V_2 = sV_1$

$-sV_2 = 3V_1 - V_{in} \longrightarrow V_1 = V_{in}\frac{1}{s^2+3}$

$V_2 = V_{in}\frac{s}{s^2+3}$

$$i_o = V_{in}\left[\frac{-8}{s^2+3} + \frac{2s}{s^2+3} + 3\right]$$

$$T(s) = \frac{i_o}{V_{in}} = \frac{3s^2+2s+1}{s^2+3}$$

$$y(s) = \frac{3s^2+2s+1}{s^2+3}$$

Check PR:

$$y(s) = \frac{3s^2 + 2s + 1}{s^2 + 3}$$

- poles : $s = \pm j\sqrt{3}$

$$\underset{s = +j\sqrt{3}}{\text{Residue}}\left[y(s)\right] = \frac{-9 + 1 + j2\sqrt{3}}{j2\sqrt{3}} = 1 + \frac{j4}{\sqrt{3}}$$

$$\underset{s = -j\sqrt{3}}{\text{Residue}}\left[y(s)\right] = \frac{-9 + 1 - j2\sqrt{3}}{-j2\sqrt{3}} = 1 - \frac{j4}{\sqrt{3}}$$

Residues of poles on $j\omega$ axis are not real $\longrightarrow$ function is <u>not</u> PR