Homework # 5

ENEE 610

Problem 1

a) Let $y(s)$ be a PR function

Check conditions 8.2-1

a) If $y(s)$ is PR, then $y(s)$ is real for real s
$\hookrightarrow \frac{1}{y(s)}$ will still also be real for real s. ✓

b) Let $y(s) = a + bs$, $\frac{1}{y(s)} = \frac{1}{a+bs}$

$$\frac{1}{y(j\omega)} = \frac{1}{a + \sigma b + j\omega b} \qquad s = \sigma + j\omega$$

$$= \frac{(a+\sigma b) - j\omega b}{(a+\sigma b)^2 + (\omega b)^2}$$

$$Re\left[\frac{1}{y(j\omega)}\right] = \frac{a + \sigma b}{(a+\sigma b)^2 + (\omega b)^2} \geq 0 \text{ for } \sigma \geq 0 \quad \text{if } a + \sigma b \geq 0$$

- since $y(s)$ is assumed PR, then condition is satisfied ✓

$\therefore$ $y(s)$ and $\frac{1}{y(s)}$ are PR which means $y\left(\frac{1}{y(s)}\right)$ will also be PR.

b) Two paths:

i) Prove $y_1\left(\frac{1}{y_2(s)}\right)$ is PR using given functions

OR ii) Evaluate $y_1\left(\frac{1}{y_2(s)}\right)$ to obtain function.

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i) From (a), if $y_1(s)$ and $y_2(s)$ are PR, then $y_1\left(\frac{1}{y_2(s)}\right)$ is PR. Therefore, we need to prove $y_1(s)$ and $y_2(s)$ are PR.

$$y_1(s) = \frac{3s}{s^2+2} + 5 = \frac{5s^2+3s+10}{s^2+2} \qquad y_2(s) = \frac{4s}{s^2+1} + 3 = \frac{3s^2+4s+3}{s^2+1}$$

$y_1(s)$

Check 3 conditions (8.2-2)

a) poles of $y_1(s)$ at $s_{1,2} = \pm j\sqrt{2} \longrightarrow$ no poles in RHP ✓

b) • poles are simple ✓

• Residues $= 3/2 \longrightarrow$ real and positive residues ✓

c) $Re[y_1(j\omega)] = Re\left[\frac{-5\omega^2+10+j3\omega}{-\omega^2+2}\right] = \frac{5(2-\omega^2)}{2-\omega^2} = 5 \longrightarrow Re[y(j\omega)] \geq 0$ for $0 \leq \omega \leq \infty$ ✓

$\therefore$ $y_1(s)$ is PR

$y_2(s)$

a) poles of $y_2(s)$ at $s_{1,2} = \pm j \longrightarrow$ no poles in RHP ✓

b) • poles are simple ✓

• Residues $= 2 \longrightarrow$ real and positive residues ✓

c) $Re[y_2(j\omega)] = Re\left[\frac{-3\omega^2+3+j4\omega}{-\omega^2+1}\right] = \frac{3(1-\omega^2)}{1-\omega^2} = 3 \longrightarrow Re[y_2(j\omega)] \geq 0$ for $0 \leq \omega \leq \infty$

$\therefore$ $y_2(s)$ is PR

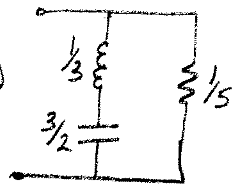
$\Rightarrow \therefore y_1\left(\frac{1}{y_2(s)}\right)$ is also PR

ii). $\frac{1}{y_2(s)} = Z_2(s) = \frac{s^2+1}{3s^2+4s+3}$

$$y_1(Z_2(s)) = \frac{3\left[\frac{s^2+1}{3s^2+4s+3}\right]}{\left[\frac{s^2+1}{3s^2+4s+3}\right]^2 + 2} \Longrightarrow y_1(Z_2(s)) = \frac{104}{9} - \frac{12s(17s^2+24s+17)}{19(19s^4+48s^3+70s+48s+19)}$$

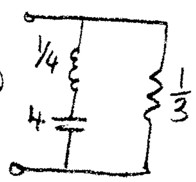
c) Synthesize $y_1(s) = \frac{3s}{s^2+2} + 5 = \frac{1}{\frac{s}{3} + \frac{2}{3s}} + 5 = \frac{1}{\left[sL + \frac{1}{sC}\right]} + \frac{1}{R}$

$y_1(s)$



Synthesize $y_2(s) = \frac{4s}{s^2+1} + 3 = \frac{1}{\frac{s}{4} + \frac{1}{4s}} + 3 = \frac{1}{\left[sL + \frac{1}{sC}\right]} + \frac{1}{R}$

$y_2(s)$



Synthesize $y_1\left(\frac{1}{y_2(s)}\right) = \frac{1}{\frac{\left(\frac{1}{y_2(s)}\right)}{3} + \frac{2}{\left(\frac{1}{y_2(s)}\right)3}} + 5 = \frac{1}{\frac{1}{3y_2(s)} + \frac{2y_2(s)}{3}} + 5 = \frac{1}{\underbrace{\frac{1}{9 + \frac{1}{\frac{s}{12} + \frac{1}{12s}}}}_{Z_a} + \underbrace{2}_{Z_b} + \underbrace{\frac{1}{\frac{3s}{8} + \frac{3}{8s}}}_{Z_c}} + \underbrace{5}_{1/Z_d}$

$= \frac{1}{Z_a + Z_b + Z_c} + \frac{1}{Z_d}$

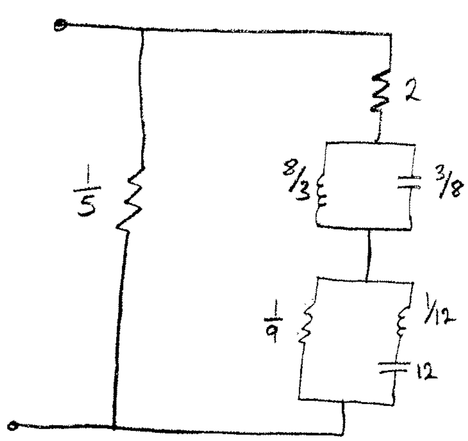
With $\frac{1}{Z_a} = 9 + \frac{1}{\frac{s}{12} + \frac{1}{12s}} = \underbrace{\frac{1}{Z_{a_1}} + \frac{1}{Z_{a_2}}}_{\text{parallel}} \rightsquigarrow Z_{a_2} = \frac{s}{12} + \frac{1}{12s} = sL + \frac{1}{sC}$ (Series)

$Z_b = 2$

$\frac{1}{Z_c} = \frac{3s}{8} + \frac{3}{8s} = \left(\frac{1}{sC} \parallel sL\right)$ (parallel)

∴

$y_1\left(\frac{1}{y_2(s)}\right)$



Problem 2

$$y(s) = \frac{s(s^2+as+b)}{(s^2+4)(s+c)}$$

First consider $c=0$:

$$y(s) = \frac{s^2+as+b}{s^2+4}$$

Check conditions (8.2-2)

- poles of $y(s) \Rightarrow s_{1,2} = \pm j2 \longrightarrow$ no pole in RHP
- Zeros of $y(s) \Rightarrow s_{1,2} = \frac{-a}{2} \pm \frac{\sqrt{a^2-4b}}{2}$

  $\hookrightarrow$ for $a^2-4b<0 \longrightarrow \frac{-a}{2} \leq 0 \longrightarrow \boxed{a \geq 0}$

  $\hookrightarrow$ for $a^2-4b>0 \longrightarrow -a \pm \sqrt{a^2-4b} \leq 0 \longrightarrow b \geq 0$

- $y(j\omega) = \frac{-\omega^2+b-j\omega a}{-\omega^2+4}$

  $Re[y(j\omega)] = 1 - \frac{b-4}{\omega^2-4} \geq 0 \longrightarrow$ satisfied for all ω only for $\boxed{b=4}$

  $\therefore$ for $\boxed{c=0,\ a\geq 0,\ b=4}$

Consider $c \neq 0$

$$y(s) = \frac{s^3+as^2+bs}{(s^2+4)(s+c)}$$

Check Conditions

- poles at $s_{1,2} = \pm j2$, $s_3 = -c$ $\longrightarrow$ if $c = \sigma + j\theta \longrightarrow Re(c) \geq 0$

  $\longrightarrow$ if c is real $\longrightarrow \boxed{c>0}$
- Zeros at $s_{1,2} = \frac{-a}{2} \pm \frac{\sqrt{a^2-4b}}{2}, 0$ $\longrightarrow a\geq 0, b \geq 0$ [as above]
- $y(j\omega) = \frac{-j\omega^3 - a\omega^2 + j\omega b}{(-\omega^2+4)(j\omega+c)}$

$$Re[y(j\omega)] = \frac{\omega^2(\omega^2-b+ac)}{(\omega^2-4)(\omega^2+c^2)} \geq 0 \longrightarrow \omega^2 - b + ac \geq 0$$

Poles on $j\omega$ axis : $s_{1,2} = \pm j2$

$$\underset{s=+j2}{\text{Residue}}\left[y(s)\right] = \frac{-4a + j(2b-8)}{j4c-8} = \frac{8(4a+bc-4c) + j16(ac-b+4)}{64+16c^2}$$

$$\underset{s=-j2}{\text{Residue}}\left[y(s)\right] = \frac{-4a + j(8-2b)}{-j4c-8} = \frac{8(4a+bc-4c) + j16(b-ac-4)}{64+16c^2}$$

<u>for real residues:</u>

$ac - b + 4 = 0$

$b - ac - 4 = 0$

<u>for positive residues</u>

$4a + bc - 4c > 0$

$\therefore \boxed{b=4 \text{ and either } a=0 \text{ or } c=0}$

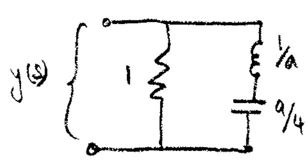
<u>Summarizing both results</u>

- if $c=0$, $b=4$ $a \geq 0$
- if $c>0$, $b=4$ $a=0$

b) <u>Synthesize</u>

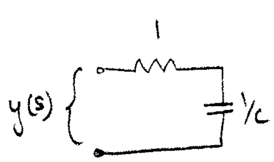
1) $c=0$ $b=4$ $\longrightarrow$ $y(s) = \dfrac{s^2+as+4}{s^2+4} = 1 + \dfrac{as}{s^2+4} = 1 + \dfrac{1}{\frac{s}{a} + \frac{4}{as}}$

$\therefore$



2) $c>0$ $b=4$ $a=0$ $\longrightarrow$ $y(s) = \dfrac{s(s^2+4)}{(s^2+4)(s+c)} = \dfrac{s}{s+c} = \dfrac{1}{1+\frac{c}{s}}$

$\therefore$



c) For non-PR with $c=0$, b, a could range and we may have more complicated synthesis since s^2+b term would not be simplified by s^2+4 in denominator.

Problem 3

a) $$y(s) = \frac{2s(s^2+4)}{s^2+2} = \frac{2s^3+8s}{s^2+2}$$

$$\sum_{\text{Ven}}[y(s)] = \frac{2s^3+8s}{s^2+2} + \frac{-2s^3-8s}{s^2+2} \equiv 0 \longrightarrow \text{can use any } k, \text{ choose } k=1$$

$$\boxed{\begin{array}{l} g = y(k) = \dfrac{2+8}{1+2} = \dfrac{10}{3} \\[2ex] C = \dfrac{g}{k} = 10/3 \end{array}}$$

$$\frac{y_{L}(s)}{y(k)} = \frac{k\,y(k) - s\,y(s)}{k\,y(s) - s\,y(k)} = \frac{\frac{10}{3} - \frac{2s^4+8s^2}{s^2+2}}{\frac{2s^3+8s}{s^2+2} - \frac{10}{3}s} = \frac{3s^2+10}{2s}$$

$$\therefore\ y_{L}(s) = y(k)\,\frac{3s^2+10}{2s}$$

$$y_{L}(s) = \frac{15s^2+50}{3s}$$

Repeat process:

$$\sum_{\text{Ven}}[y_{L}(s)] = \frac{15s^2+50}{3s} - \frac{15s^2+50}{3s} \equiv 0 \longrightarrow \text{can use any } k, \text{ choose } k=1$$

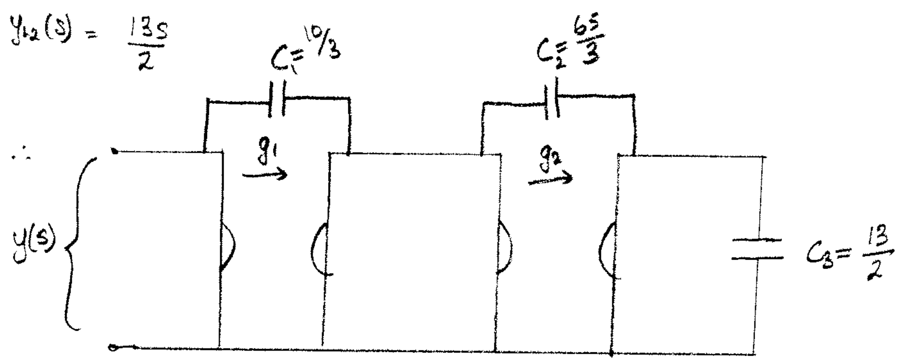
$$\frac{y_{L2}(s)}{y_{L}(k)} = \frac{k\,y_{L}(k) - s\,y_{L}(s)}{k\,y_{L}(s) - s\,y_{L}(k)} = \frac{\frac{65}{3} - \frac{15s^2+50}{3}}{\frac{15s^2+50}{3s} - \frac{65s}{3}} = \frac{3s}{10}$$

$$\boxed{\begin{array}{l} g = y(k) = \dfrac{15+50}{3} = \dfrac{65}{3} \\[2ex] C = g/k = \dfrac{65}{3} \end{array}}$$

$$\therefore\ y_{L2}(s) = y_{L}(k)\,\frac{3s}{10}$$

$$y_{L2}(s) = \frac{13s}{2}$$

$\therefore$



$g_1 = \frac{10}{3}$

$g_2 = \frac{65}{3}$

Using different 'k' values

- We showed that the even part of $y(s) = 0$ for all k, so we can choose any k
- If k is increased the form of the resulting $y_L(s)$ remains, but the g and C values will increase :

$k = 1 \quad g = 10/3 \quad C = 10/3$

$k = 2 \quad g = 16/3 \quad C = 8/3$

$k = 4 \quad g = \frac{80}{9} \quad C = \frac{20}{9}$

- If k is negative, we will see the same resulting $y_L(s)$ as the corresponding positive k, but gyrator constant will be flipped:

$k = -1 \quad g = -10/3 \quad C = 10/3$

$k = -2 \quad g = -16/3 \quad C = 8/3$

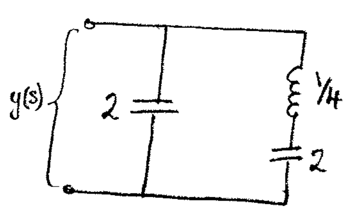
$k = -4 \quad g = \frac{-80}{9} \quad C = 20/9$

b) 1st Cauer

$y(s) \Rightarrow$

$$\begin{array}{l} s^2+2 \,\big)\, 2s^3 + 8s \quad (2s \\ \quad\;\; 2s^3 + 4s \\ \quad\;\; \overline{4s \,\big)\, s^2 + 2 \quad (\tfrac{1}{4}s} \\ \qquad\quad\; s^2 \\ \qquad\quad\; \overline{2 \,\big)\, 4s \quad (2s} \\ \qquad\qquad 4s \\ \qquad\qquad \overline{0} \end{array}$$

$$y(s) = 2s + \cfrac{1}{\tfrac{1}{4}s + \cfrac{1}{2s}}$$

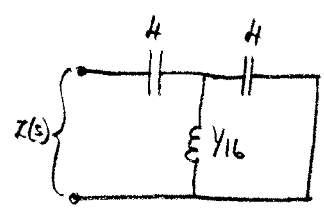


2nd Cauer

$Z(s) \Rightarrow$

$$\begin{array}{l} 8s + 2s^3 \,\big)\, 2 + s^2 \quad (\tfrac{1}{4s} \\ \quad\;\; 2 + \tfrac{1}{2}s^2 \\ \quad\;\; \overline{\tfrac{1}{2}s^2 \,\big)\, 8s + 2s^3 \quad (16/s} \\ \qquad\quad 8s \\ \qquad\quad \overline{2s^3 \,\big)\, \tfrac{1}{2}s^2 \quad (\tfrac{1}{4s}} \\ \qquad\qquad \tfrac{1}{2}s^2 \\ \qquad\qquad \overline{0} \end{array}$$

$$Z(s) = \frac{1}{4s} + \cfrac{1}{\cfrac{16}{s} + \cfrac{1}{1/4s}}$$



1st Foster

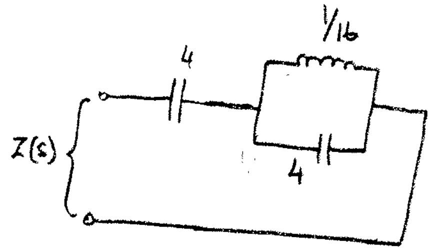
$$\frac{1}{y(s)} = \frac{s^2+2}{2s^3+8s} = 0 + \frac{k_0}{2s} + \frac{k_1 s}{s^2+4}$$

$$s^2+2 = k_0(s^2+4) + 2k_1 s^2$$

$$s^2=0 \Rightarrow \quad k_0 = 1/2$$

$$s^2=-4 \Rightarrow \quad k_1 = 1/4$$

$$Z(s) = \frac{1}{y(s)} = \frac{1}{4s} + \frac{s}{4s^2+16} = \frac{1}{4s} + \frac{1}{4s + \frac{16}{s}}$$



2nd Foster

$$y(s) = \frac{2s^3+8s}{s^2+2} = k_0 s + \frac{k_1 s}{s^2+2} = 2s + \frac{4s}{s^2+2} = 2s + \frac{1}{\frac{s}{4} + \frac{1}{2s}}$$