Homework # 2

ENEE 610

Problem # 1

a)

$$E = \begin{bmatrix} 0 & 0 & 0 & 0 \\ C_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & L_1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \longrightarrow \hat{E} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

- Use permutation matrices to perform this transformation:

$$T_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \qquad T_2 = \begin{bmatrix} 0 & \frac{1}{C_1} & 0 & 0 \\ 0 & 0 & 1/L_1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Then:

$$E\dot{x} = Ax + Bu \qquad\qquad y = Cx$$

$$\Downarrow \qquad\qquad\qquad \Downarrow$$

$$\underbrace{(T_2 E T_1)}_{\hat{E}}\underbrace{(T_1^{-1}\dot{x})}_{\dot{\hat{x}}} = \underbrace{(T_2 A T_1)}_{\hat{A}}\underbrace{(T_1^{-1}x)}_{\hat{x}} + \underbrace{T_2 B}_{\hat{B}} u \qquad\qquad y = \underbrace{C T_1}_{\hat{C}} \underbrace{T_1^{-1} x}_{\hat{x}}$$

- Performing the matrix calculations, we obtain:

$$\hat{E}\dot{\hat{x}} = \hat{A}\hat{x} + \hat{B}u \qquad\qquad y = \hat{C}\hat{x}$$

$$\left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \left[\begin{array}{c} \dot{x}_1 \\ x_4 \\ \hline x_3 \\ x_2 \end{array}\right] = \left[\begin{array}{cc|cc} -\frac{g_1}{C_1} & 0 & \frac{1}{C_1} & 0 \\ 0 & 1/L_1 & 0 & 0 \\ \hline 0 & 0 & 0 & g_1 \\ g_1 & 0 & -1 & 0 \end{array}\right] \left[\begin{array}{c} x_1 \\ x_4 \\ \hline x_3 \\ x_2 \end{array}\right] + \left[\begin{array}{cc} 0 & \frac{1}{C_1} \\ \frac{1}{L_1} & 0 \\ \hline 0 & 0 \\ 0 & 0 \end{array}\right] \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right] \left[\begin{array}{c} x_1 \\ x_4 \\ \hline x_3 \\ x_2 \end{array}\right]$$

b) Partition matrices:

$$\begin{bmatrix} I_2 & 0_2 \\ 0_2 & 0_2 \end{bmatrix} \begin{bmatrix} \dot{\hat{x}}_1 \\ \hat{x}_2 \end{bmatrix} = \left[\begin{array}{c|c} \hat{A}_{11} & \hat{A}_{12} \\ \hline \hat{A}_{21} & \hat{A}_{22} \end{array}\right] \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} + \left[\begin{array}{c} \hat{B}_1 \\ \hline \hat{B}_2 \end{array}\right] u \qquad\qquad \hat{y} = [\underbrace{\hat{C}_1}_{C_s}\ \underbrace{\hat{C}_2}_{0_2}] \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix}$$

- from 2^{nd} row:

$$0 = \hat{A}_{21}\hat{x}_1 + \hat{A}_{22}\hat{x}_2 + \hat{B}_2 u \longrightarrow \hat{x}_2 = -\hat{A}_{22}^{-1}\hat{A}_{21}\hat{x}_1 - \hat{A}_{22}^{-1}\hat{B}_2 u \quad (1)$$

- from 1^{st} row:

$$\dot{\hat{x}}_1 = \hat{A}_{11}x_1 + \hat{A}_{12}x_2 + \hat{B}_1 u$$

- substitute (1):

$$\dot{\hat{x}}_1 = \hat{A}_{11}x_1 + A_{12}\left[-\hat{A}_{22}^{-1}\hat{A}_{21}\hat{x}_1 - \hat{A}_{22}^{-1}\hat{B}_2 u\right] + \hat{B}_1 u \longrightarrow \dot{\hat{x}}_1 = \underbrace{\left[A_{11} - A_{12}A_{22}^{-1}A_{21}\right]}_{A_s}\hat{x}_1 + \underbrace{\left[B_1 - \hat{A}_{12}\hat{A}_{22}\hat{B}_2\right]}_{B_s} u$$

Performing matrix calculations, we obtain:

$$\dot{\hat{x}}_1 = \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 1/L_1 \end{bmatrix}}_{A_s} \hat{x}_1 + \underbrace{\begin{bmatrix} 1/C_1 & 1/C_1 \\ 1/L_1 & 0 \end{bmatrix}}_{B_s} u \qquad \hat{y} = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{C_s} \hat{x}_1 + \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}}_{D_s} u$$

[* Alternatively: expand equations from result of part (a), solve for two variables, and write matrix equation]

c) From given semi-state matrices A, B, C, E:

$$T(s) = C \cdot (sE - A)^{-1} \cdot B$$

$$= \begin{bmatrix} \frac{1}{sC_1} & \frac{1}{sC_1} \\ \frac{1}{sL_1 - 1} & 0 \end{bmatrix}$$

From calculated state variable matrices A_s, B_s, C_s, D_s:

$$T(s) = C_s \cdot (sI_2 - A_s)^{-1} \cdot B_s + D_s$$

$$= \begin{bmatrix} \frac{1}{sC_1} & \frac{1}{sC_1} \\ \frac{1}{sL_1 - 1} & 0 \end{bmatrix}$$

$\Longrightarrow$ the two transfer functions match.