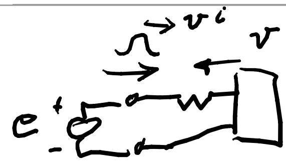


$$v^n = S(s)\,v^i \quad ; \quad \underbrace{1 - S^T(-j\omega)S(j\omega)}_{\|S(j\omega)\|^2 \le 1} \ge 0$$

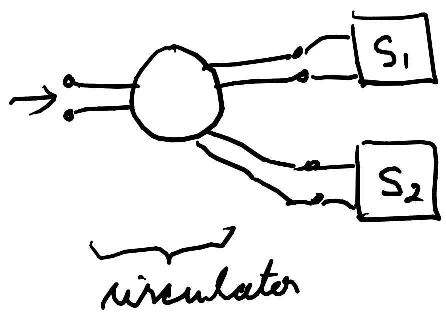
S 1) no singularities in $\sigma \ge 0$ if rational
2) if rational real coefficients
3) $\|S(j\omega)\| \le 1$

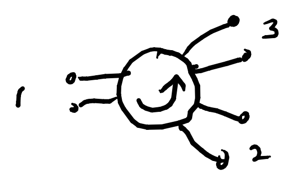
} BR rational bounded real

Note if $S(s) = S_1(s)\cdot S_2(s)$ & S_1 & S_2 are BR then $S(s)$ is BR

as $\|S(j\omega)\| = \|S_1(j\omega)\| \cdot \|S_2(j\omega)\| \le 1 \cdot 1 \le 1$

Means we can synthesize by factoring; uses a circulator

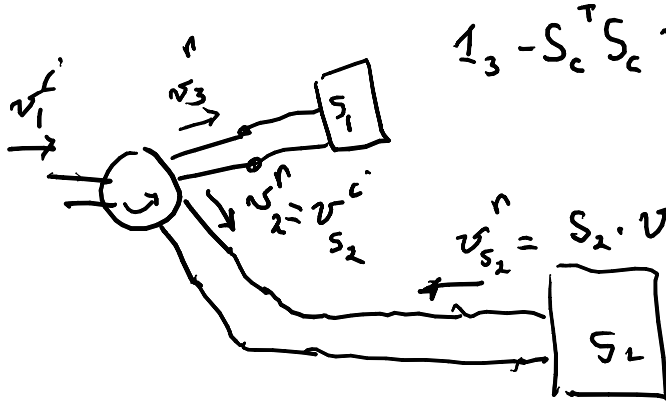
$S = S_1 S_2 \rightarrow$ 



$$\begin{bmatrix} v_1^n \\ v_2^n \\ v_3^n \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} v_1^i \\ v_2^i \\ v_3^i \end{bmatrix}$$

$$S_c = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad ; \quad S_c^T S_c = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$1_3 - S_c^T S_c = 0_3 \Rightarrow S_c \text{ is BR}$$



$$v_{S_2}^n = S_2 \cdot v_{S_2}^i = v_2^i = v_3^n = v_{S_1}^i$$

$$v_{S_1}^n = S_1 v_{S_1}^i = S_1 \cdot S_2 v_{S_2}^i$$
$$= S_1 S_2 v_2^n = S_1 S_2 v_1^i$$

$$v_3^i = v_{S_1}^n = \underbrace{S_1 S_2 v_1^i = v_1^n}_{} = S \cdot v_1^i$$

$$S = S_1 \cdot S_2$$

Look at Richards' function in this light

$$y_L(s) = y(k)\left[\frac{k\,y(k) - s\,y(s)}{k\,y(s) - s\,y(k)}\right]$$

$$S = \frac{1-y}{1+y} \Rightarrow \quad S_L(s) = \frac{1 - y_L(s)/y(k)}{1 + y_L(s)/y(k)} = \frac{k\,y(s) - s\,y(k) - k\,y(k) + s\,y(s)}{k\,y(s) - s\,y(k) + k\,y(k) - s\,y(s)}$$

$$= \frac{k(y(s) - y(k)) - s(y(k) - y(s))}{k(y(s) + y(k)) - s(y(k) + y(s))}$$

$$= \frac{(k+s)(y(s) - y(k))}{(k-s)(y(s) + y(k))} = \left(\frac{k+s}{k-s}\right)\left(\frac{y(s)/y(k) - 1}{y(s)/y(k) + 1}\right)$$

$$= \left(\frac{k+s}{k-s}\right)\cdot\left(\frac{\frac{y(s)}{y(k)} - 1}{\frac{y(s)}{y(k)} + 1}\right); \quad S = \frac{z-1}{z+1}$$

no poles in $\sigma \geq 0$

if $y(s)$ is PR & 2nd term is BR, if k is real has real coefficients & $\left|\frac{y(s) - y(k)}{y(s) + y(k)}\right|_{s=j\omega} \leq 1$ by BR property

and $\left|\frac{k + j\omega}{k - j\omega}\right| = \frac{\sqrt{k^2+\omega^2}}{\sqrt{k^2+\omega^2}} = 1$ if k is real

$$\Rightarrow |S_L(j\omega)| = \left|\frac{k+j\omega}{k-j\omega}\right| \cdot |S(j\omega)| \leq 1 \cdot 1 \Rightarrow S_L(s) \text{ is BR}$$

$\Rightarrow y_L(s)$ is PR

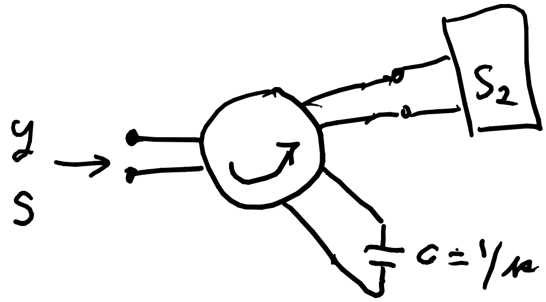
if k is real & >0

Note $S(s) = \frac{k-s}{k+s} \cdot S_L(s)$ if k is real & >0

$$= \frac{1 - s/k}{1 + s/k} \cdot S_L(s)$$

$$\frac{1 - s/k}{1 + s/k} = \frac{1-y}{1+y} \Rightarrow y = s/k$$

$C = 1/k$



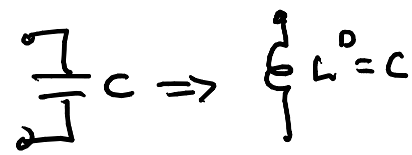
$$S_2 = \frac{1 - y_R(s)/y(s)}{1 + y_R(s)/y(s)} = S_2(s)$$

$y_R =$ Richards' function

$$\frac{y_R}{y(k)}$$

needs $k =$ zero of $y(s) + y(-s)$ for S_2 of smaller degree than S ($=$ degree of $y(s)$)

Dual: $y(s)$; $y^D(s) = z(s) = \dfrac{1}{y(s)}$



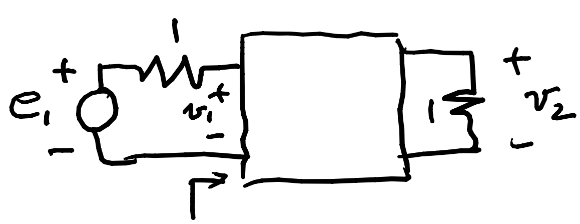
$$S^D(s) = \frac{1 - y^D(s)}{1 + y^D(s)} \therefore$$

$$S(s) = \frac{1 - y(s)}{1 + y(s)} = \frac{1 - 1/z(s)}{1 + 1/z(s)} = \frac{z(s) - 1}{z(s) + 1} = \frac{y^D(s) - 1}{y^D(s) + 1}$$

$$= -\left(\frac{1 - y^D(s)}{1 + y^D(s)}\right) = -S^D(s)$$

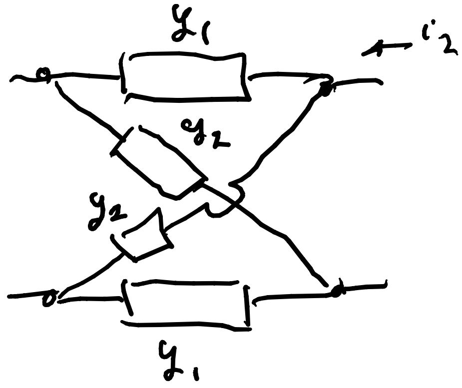
$$S^D(s) = -S(s)$$

Constant R:



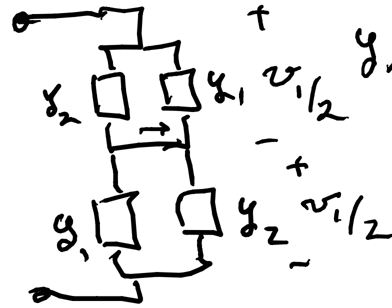
$z_{in} = y_{in} = 1 \Rightarrow$

Lattice



$y_{22} = y_{11}$, $y_{12} = y_{21}$

For $y_{11} = \frac{i_1}{v_1}\Big|_{v_2=0=\text{short}}$



$$y_{in} = \frac{1}{2}(y_1+y_2)$$
$$= y_{11} = y_{22}$$

$y_{21} = \frac{i_2}{v_1}\Big|_{v_2=0=\text{short}}$

$$i_2 = y_2\left(\frac{v_1}{2}\right) - y_1(v_1/2)$$
$$= \frac{1}{2}(y_2 - y_1)\, v_1$$

$$y_{21} = \frac{1}{2}(y_2 - y_1) = y_{12}$$

$$Y = \frac{1}{2}\begin{bmatrix} y_2+y_1 & y_2-y_1 \\ y_2-y_1 & y_2+y_1 \end{bmatrix}$$

$$\begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$-y_L v_2 = i_2$

$$y_{in} = y_{11} - \frac{y_{12}\,y_{21}}{y_L + y_{22}}$$
$$= \frac{\Delta y + y_L y_{11}}{y_L + y_{22}}$$

for $y_L = 1\,℧$

$$\Rightarrow y_{in} = \frac{\Delta y + y_{11}}{1 + y_{22}}$$

put y_L on a lattice

$$\Delta y = \frac{1}{4}\left[(y_2+y_1)^2 - (y_2-y_1)^2\right] = \frac{1}{4}\, 4\, y_2 y_1 = y_2 y_1$$

$$y_{in} = \frac{y_2 y_1 + \frac{1}{2}(y_2+y_1)}{1 + \frac{1}{2}(y_2+y_1)} = 1 \text{ for constant R lattice}$$

$$\Rightarrow\; 2 y_2 y_1 + \cancel{y_2} + \cancel{y_1} = 2 + \cancel{y_2} + \cancel{y_1} \;\Rightarrow\; y_2 = \frac{1}{y_1} = y_1^D = z_1$$

$$\therefore \quad Y_{\substack{\text{const } R\\ \text{lattice}}} = \frac{1}{2}\begin{bmatrix} y_1 + \frac{1}{y_1} & \frac{1}{y_1} - y_1 \\ \frac{1}{y_1} - y_1 & y_1 + \frac{1}{y_1} \end{bmatrix}$$

$$v_1 = (1/2)e_1 \ , \quad \frac{v_2}{e_1} = \frac{v_2}{2v_1}$$

$$\text{for } v_2 = -1 \cdot i_2 = -1\left(\frac{1}{2}\left(\frac{1}{y_1} - y_1\right)v_1 + \frac{1}{2}\left[y_1 + \frac{1}{y_1}\right)v_2\right)$$

$$\left(1 + \frac{1}{2}\left(y_1 + \frac{1}{y_1}\right)\right)v_2 = -\frac{1}{2}\left(\frac{1}{y_1} - y_1\right)v_1$$

$$\frac{v_2}{v_1} = \frac{(y_1 - 1/y_1)}{2 + \left(y_1 + \frac{1}{y_1}\right)} = \frac{y_1^2 - 1}{y_1^2 + 2y_1 + 1} = \frac{(y_1 - 1)(y_1 + 1)}{(y_1 + 1)^2}$$

$$= \frac{y_1 - 1}{y_1 + 1} = -S_1 = 2 \cdot \frac{v_2}{e_1} \Rightarrow \frac{v_2}{e_1} = \frac{1}{2}\,\frac{y_1 - 1}{y_1 + 1}$$

If $y_1 \Rightarrow$ PR lossless then for $y_1(s) = \frac{n(s)}{d(s)} \Rightarrow y_1 + 1 = \frac{n_1 + d_1}{d_1}$

and $n_1 + d_1 =$ Hurwitz polynomial