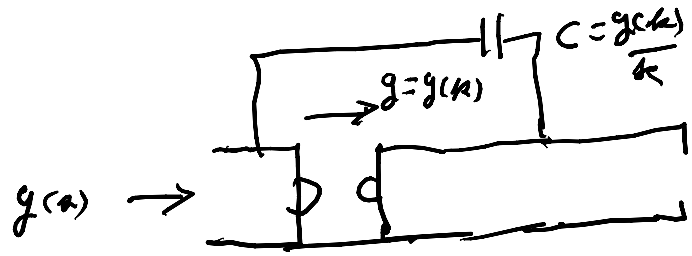



$$y_L = g\,G(s) = y(k)\cdot\frac{k\,y(k) - s\,y(s)}{k\,y(s) - s\,y(k)}$$

## Positive Real functions p. 338

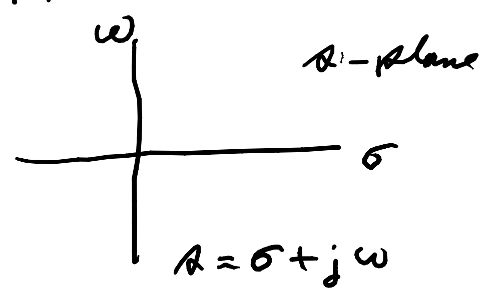
PR = rational in $s$

$f(s)$

1. $f(s)$ is real for $s$ real & positive (i.e. for $\sigma > 0$) (a real circuit)

2. $f(s)$ is analytic in $\sigma > 0$ (stable circuit)

3. $\operatorname{Re} f(s) \geq 0$ in $\sigma > 0$ (a passive circuit)




Ex: $f(s) = \dfrac{as + b}{s+1}$ $a$ & $b$ real by 1. pole at $s = -1$ in $\sigma < 0$

$$\operatorname{Re} f(\sigma + j\omega) = \operatorname{Re}\left(\frac{a[\sigma + j\omega] + b}{\sigma + j\omega + 1}\right) = \operatorname{Re}\left[\frac{(a\sigma + b + aj\omega)((1+\sigma) - j\omega)}{[(1+\sigma) + j\omega][(1+\sigma) - j\omega]}\right]$$

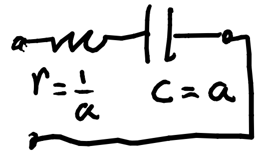
$$= \operatorname{Re}\frac{\{(a\sigma + b)(1+\sigma) + a\omega^2 + j\omega\,(a(1+\sigma) - (a\sigma + b))\}}{(1+\sigma)^2 + \omega^2}$$

$$= \frac{a\sigma + a\sigma^2 + b + b\sigma + a\omega^2}{(1+\sigma)^2 + \omega^2}$$ check $b + a\sigma + a\sigma^2 + b\sigma + a\omega^2 \geq 0$ in $\sigma > 0$

for $\omega = 0$ & $\sigma \approx 0$ but $> 0 \Rightarrow b \geq 0$ ; if $b = 0 \Rightarrow a(\sigma + \sigma^2) \geq 0$ $\Rightarrow a \geq 0$

if $b = 0$, $a > 0$ then $f(s) = \dfrac{as}{s+1} \Rightarrow$ if $y(s) = f(s) = \dfrac{1}{1/a + 1/as}$

$\Rightarrow z(s) = \frac{1}{a} + \frac{1}{as} \Rightarrow$

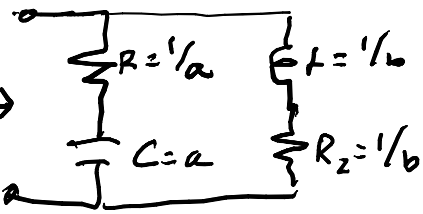



if $b>0$ take $\sigma \to \infty$ $\quad a\sigma + a\sigma^2 + b\sigma \sim a\sigma^2 \Rightarrow a \geq 0$

$\Rightarrow$ $f(s)$ is PR if $a \geq 0$ & $b \geq 0$

if $a>0$ & $b>0$ then $y(s) = \frac{as+b}{s+1} = \frac{as}{s+1} + \frac{b}{s+1}$

$$= \frac{1}{\frac{1}{a} + \frac{1}{as}} + \frac{1}{\frac{s}{b} + \frac{1}{b}}$$

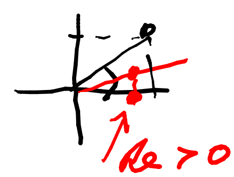



Ex: $y(s) = \sqrt{s}$ condition 1. $\sqrt{\sigma}$ is real if $\sigma > 0$

2. singularity; no derivative at $s=0$; ok in $\sigma>0$

3. $\mathrm{Re}\sqrt{(\sigma+j\omega)}$ in $\sigma > 0$

$$\mathrm{Re}\left(|\sigma^2+\omega^2|^{1/2} e^{j\frac{\tan^{-1}\omega/\sigma}{2}}\right) = \cos\left(\frac{\tan^{-1}(\omega/\sigma)}{2}\right) > 0 \text{ in } \sigma > 0$$

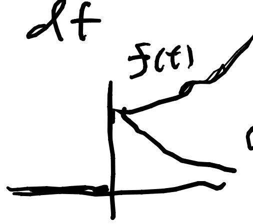



the circuit is an infinitely long RC transmission, $R=1, C=1$ per unit length

Laplace transform (bilateral)

$$\mathcal{L}[f(t)] = \int_{-\infty}^{\infty} f(t) e^{-st}\,dt$$

Ex: $f(t) = e^{at} 1(t)$

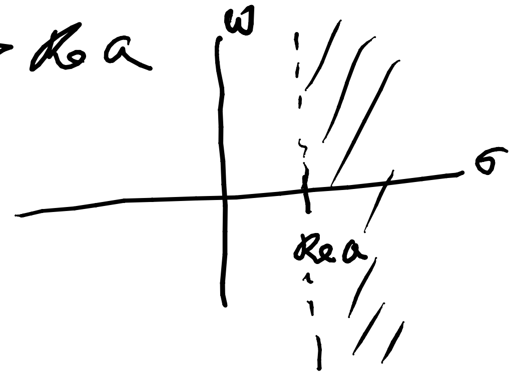



$1(t) = \begin{cases} 1 & t>0 \\ 0 & t<0 \end{cases}$

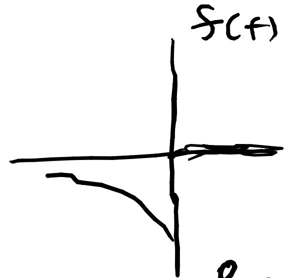
$$\mathcal{L}[e^{at}1(t)] = \int_{-\infty}^{\infty} e^{at} 1(t) e^{-st}\,dt = \int_0^{\infty} e^{(a-s)t}\,dt$$

$$= \frac{1}{(a-s)} e^{(a-s)t}\Big|_{t=0}^{t=\infty} = -\frac{1}{a-s} + \frac{1}{a-s} e^{(a-s)(+\infty)} \rightarrow 0 \text{ if } \mathrm{Re}\, s > \mathrm{Re}\, a$$

$\mathcal{L}[e^{at}1(t)] = \frac{1}{s-a}$ if $\mathrm{Re}\, s = \sigma > \mathrm{Re}\, a$




Ex: $\mathcal{L}[-e^{at}1(-t)]$

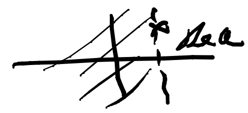



$$= \int_{-\infty}^{\infty} -e^{at}1(-t)e^{-st}\,dt = -\int_{-\infty}^{0} e^{(a-s)t}\,dt = \frac{-1}{a-s}e^{(a-s)t}\Big|_{t=-\infty}^{0}$$

$$= \frac{-1}{a-s} + \frac{-e^{(a-s)t}}{a-s}\Big|_{t=-\infty}$$

$\quad\quad\quad\quad\quad\quad\quad\quad \overset{\shortparallel}{0} \Rightarrow \mathrm{Re}\,(a-s)(-1) < 0 \Rightarrow \mathrm{Re}\, s < \mathrm{Re}\, a$

$\mathcal{L}[-e^{at}1(-t)] = \frac{1}{s-a}$ in $\mathrm{Re}\, s < \mathrm{Re}\, a$




$\mathcal{L}[e^{at}1(t)] = \frac{1}{s-a}$ in $\mathrm{Re}\, s > \mathrm{Re}\, a$

Unit step response: $\mathcal{L}[1(t)] = \frac{1}{s}$ in $\mathrm{Re}\, s > 0$

Ex: of unit step response

$$T(s) = \frac{s-1}{s^2+4} \Rightarrow T(s) \times \frac{1}{s} = \frac{1}{s} \cdot \frac{s-1}{s^2+4}$$

$$y_u(t) = \mathcal{L}^{-1}\left[\frac{1}{s}\cdot\frac{s-1}{s^2+4}\right]; \quad \frac{1}{s}\cdot\frac{s-1}{s^2+4} = \frac{k_0}{s} + \frac{k_2}{s+j2} + \frac{k_2^*}{s-j2}$$

$$= \frac{k_0}{s} + \frac{sk_2 + sk_2^* + j2k_2^* - j2k_2}{s^2+4}$$

$$k_0 = s\left[\frac{1}{s}\cdot\frac{s-1}{s^2+4}\right]\Big|_{s=0} - \frac{sk_2}{s+j2}\Big|_{s=0} + \frac{sk_2^*}{s-j2}\Big|_{s=0} = \frac{0-1}{0^2+4} = -\frac{1}{4}$$

$$k_2 = (s+j2)\left[\frac{1}{s}\cdot\frac{s-1}{(s+j2)(s-j2)}\right]\Bigg|_{s=-j2} = \frac{1}{-j2}\cdot\frac{-1-j2}{2(-j2)} = \frac{1}{-8}(-1-j2) = \frac{1}{8}+j\frac{1}{4}$$

$$k_2^* = \frac{1}{8} - j\,1/4$$

$$\frac{1}{s}\cdot\frac{s-1}{s^2+4} = \frac{-1/4}{s} + \frac{\frac{1}{8}+j\,1/4}{s+j2} + \frac{\frac{1}{8}-j\,1/4}{s-j2} \qquad \mathrm{Re}\, s > 0$$

$$y_u(t) = -\frac{1}{4}1(t) + \left(\frac{1}{8}+j\frac{1}{4}\right)e^{-j2t}1(t) + \left(\frac{1}{8}-j\frac{1}{4}\right)e^{j2t}1(t)$$

$$= -\frac{1}{4}1(t) + \left[\frac{1}{4}\cdot\frac{\left(e^{j2t}+e^{-j2t}\right)}{2} + \frac{1}{2}\frac{\left(-e^{j2t}+e^{-j2t}\right)}{2(-j)}\right]1(t)$$

$$= -\frac{1}{4}1(t) + \frac{1}{4}\cos 2t\cdot 1(t) + \frac{1}{2}\sin 2t\cdot 1(t)$$

$$y_\delta(t) = \frac{d}{dt}y_u(t) = -\frac{1}{4}\delta(t) + \frac{1}{4}\delta(t) - \frac{1}{2}\sin 2t\cdot 1(t) + \cos 2t\cdot 1(t)$$

$$= \left(\cos 2t - \frac{1}{2}\sin 2t\right)\cdot 1(t)$$