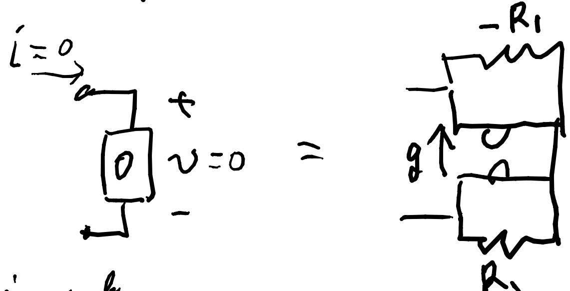
EE 610

09/13/12

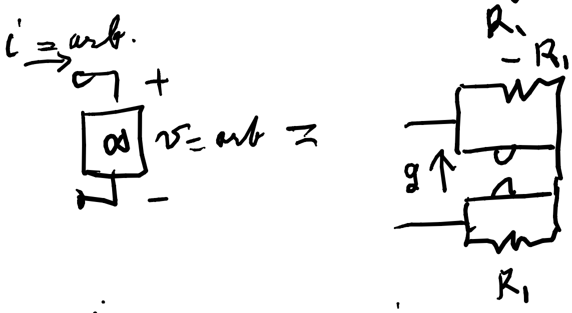
if $gR_1 = 1$ & $gR_2 = 1$ $\Rightarrow$ this is a nullator

as $v = 0$ & $i = 0$

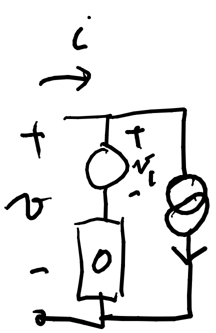
if $gR_1 = -1$ & $gR_2 = 1$ then v_2 is arbitrary $\Rightarrow$ v_1 is arbitrary & i is arbitrary

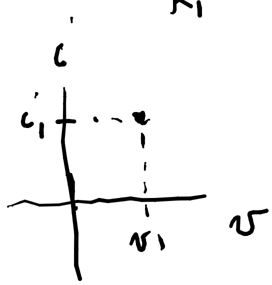


$gR_1 = 1$ nullator $\begin{bmatrix}1\\0\end{bmatrix} v = \begin{bmatrix}0\\1\end{bmatrix} i$



$gR_1 = -1$ norator $[0]\, v = [0]\, i$





$$1_k = \underbrace{\begin{bmatrix} 1 & 0 & \cdots & \\ 0 & 1 & \cdots & 0 \\ \vdots & & 0 & 1 \end{bmatrix}}_{k} \Big\} k$$

$$E\frac{dx}{dt} = Ax + Bu \qquad y = Cx$$

diagonalize E

$$\begin{bmatrix} 1_{k_1} & 0 \\ 0 & 0_{k_2} \end{bmatrix} \frac{d\bar{x}}{dt} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \bar{x} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u$$

if A_{22}^{-1} exists then $\Rightarrow$ $\begin{bmatrix} 1_{k_1} & 0 \\ 0 & 0_{k_2} \end{bmatrix} \frac{d\bar{x}}{dt} = \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{22}^{-1}A_{21} & 1_{k_2} \end{array}\right] \bar{x} + \begin{bmatrix} B_1 \\ A_{22}^{-1}B_2 \end{bmatrix} u$

$$\bar{x} = \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix}$$

$$\bar{x}_2 = -A_{22}^{-1} A_{21} \bar{x}_1 - A_{22}^{-1} B_2 u$$

into $\bar{x} \Rightarrow$ $$1_{k_1} \frac{d\bar{x}_1}{dt} = A_{11}\bar{x}_1 + A_{12}(-A_{22}^{-1}A_{21})\bar{x}_1 + (B_1 - A_{22}^{-1}B_2)u$$

$$y = [C_1 \quad C_2]\begin{bmatrix}\bar{x}_1 \\ \bar{x}_2\end{bmatrix} \simeq [C_1\bar{x}_1 + C_2\{-A_{22}^{-1}A_{21}\bar{x}_1 - A_{22}^{-1}B_2 u]$$

$$= [C_1 - C_2 A_{22}^{-1} A_{21}]\bar{x}_1 + (-C_2 A_{22}^{-1} B_2)u$$

$(+ D_1 \frac{du}{dt} + D_2 \frac{d^2u}{dt^2}$

$\to \ldots$

if A_{22} is singular but last k_2 rows of A are of rank k_2

This brings us to state-variable theory where $\bar{x}_1 =$ state

$$\frac{dx}{dt} = Ax + Bu, \quad y = Cx + Du + D_1\frac{du}{dt} \ldots$$

If $u = 0$; then we would solve $\frac{dx}{dt} = Ax$ subject to $x(0)$

Lets define the exponential of a square matrix; for $a = 1\times 1$

$$e^a = \sum_{i=0}^{\infty} a^i/i! \Rightarrow \text{for } A \; n\times n \Rightarrow e^A = \sum_{i=0}^{\infty}\frac{A^i}{i!}$$

Here use $A \to A\cdot t$; $A =$ const.

$$e^{At} = \sum_{i=0}^{\infty}(At)^i/i! \; ; \quad \frac{de^{At}}{dt} = \sum_{i=\cancel{0}\,1}\frac{A^i}{i!}\cdot i\, t^{i-1} = \sum_{i=1}^{\infty}\frac{A^{i-1}}{(i-1)!}\times A\cdot t^{i-1}$$

$$= A\cdot\sum_{i=1}^{\infty}\frac{A^{i-1}}{(i-1)!}\cdot t^{i-1}$$

let $j = i-1$
$i = 1 \Rightarrow j = 0$

$$= A\cdot\sum_{j=0}^{\infty}\frac{A^j}{j!}\cdot t^j$$

$$= A e^{At}$$

shows that $e^{At}x(0)$ satisfies $\frac{d\, e^{At}x(0)}{dt} = A \cdot e^{At}x(0)$

$$|||$$

$$\frac{dx}{dt} = Ax, \quad x(0)$$

To calculate e^{At}

$$\frac{dx}{dt} = Ax + Bu \implies s\mathbb{1}_k x = Ax + Bu; \ (s\mathbb{1}_k - A)x = Bu$$

$x = (s\mathbb{1}_k - A)^{-1} \cdot Bu$ note $s\mathbb{1}_k - A$ evaluated on

$$s = A \implies A - A = 0_k$$

$$P(s) = \det(s\mathbb{1}_k - A) = 0$$

$$P(A) = 0_k$$

Ex: $A = \begin{bmatrix} 0 & 1 \\ -a_0 & -a_1 \end{bmatrix}$; $\det(s\mathbb{1}_2 - A) = \det\begin{bmatrix} s & -1 \\ a_0 & s+a_1 \end{bmatrix}$

$= s^2 + a_1 s + a_0 = P(s)$

characteristic polynomial

$$\implies A^2 + a_1 A + a_0 \mathbb{1}_2 = 0_2$$

$$\begin{bmatrix} 0 & 1 \\ -a_0 & -a_1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -a_0 & -a_1 \end{bmatrix} = \begin{bmatrix} -a_0 & -a_1 \\ a_0 a_1 & -a_0 + a_1^2 \end{bmatrix}$$

$$a_1\begin{bmatrix} 0 & 1 \\ -a_0 & -a_1 \end{bmatrix} = \begin{bmatrix} 0 & a_1 \\ -a_0 a_1 & -a_1^2 \end{bmatrix}$$

add these 3

$$a_0\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a_0 & 0 \\ 0 & a_0 \end{bmatrix}$$

$$\begin{bmatrix} -a_0 + 0 + a_0 & -a_1 + a + 0 \\ a_0 a_1 - a_0 a_1 + 0 & -a_0 + a_1^2 - a_1^2 + a_0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\implies A^2 = -a_1 A - a_0 \mathbb{1}_2$$

$\therefore$ e^{At} here is $f_1(t) A + f_0(t) \mathbb{1}_2$

For $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, e^{At} ; $P(s) = \det(s 1_2 - A) = \det \begin{bmatrix} s & -1 \\ 1 & s \end{bmatrix} = s^2 + 1$

$$\det(s 1_2 - At) = \det \begin{bmatrix} s & -t \\ t & s \end{bmatrix} = s^2 + t^2$$

At satisfies $(At)^2 + 1_2 = A^2 t^2 + t^2 1_2 = 0_2$

$$= A^2 t^2 = -t^2 1_2$$

$$A^3 = A^2 \cdot A = -A, \quad A^4 = A^2 \cdot A^2 = (-1_2)(-1_2) = 1_2$$

$$A^5 = A^3 \cdot A^2 = A \qquad = A^{2 \cdot p} = (-1)^p 1_2 \quad p = 1, 2, \ldots \infty$$

$$A^{2 \cdot p + 1} = (-1)^p A$$

$$e^{At} = \sum_{i=0}^{\infty} \frac{A^i t^i}{i!} = \sum_{p=0}^{\infty} \frac{A^{2p} t^{2p}}{2p!} + \sum_{p=0}^{\infty} \frac{A^{2p+1}}{(2p+1)!} \cdot t^{2p+1}$$

$$= \sum_{p=0}^{\infty} \frac{(-1)^p 1_2 t^{2p}}{2p!} + \sum_{p=0}^{\infty} \frac{(-1)^p \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} t^{2p+1}}{(2p+1)!}$$

$$= \begin{bmatrix} \cos t & 0 \\ 0 & \cos t \end{bmatrix} + \begin{bmatrix} 0 & \sin t \\ -\sin t & 0 \end{bmatrix}$$

In general A satisfies the minimal polynomial which has to be a factor of the characteristic polynomial.