setting up state variable equations for a single input single output given the transfer function

EE 610
10/29/09

$$\frac{y(s)}{u(s)} = \frac{n(s)}{d(s)} = \frac{n_{\delta-1}s^{\delta-1} + \cdots + n_1 s + n_0}{s^\delta + d_{\delta-1} s^{\delta-1} + \cdots + d_1 s + d_0} = T(s)$$

$$E\dot{x} = Ax + Bu \quad ; \quad E = 1_\delta \,, \ \delta = \text{degree of } T(s) \qquad x = \begin{bmatrix} x_1 \\ \vdots \\ x_\delta \end{bmatrix}$$
$$y = Cx$$

$$T(s) = C(s1_\delta - A)^{-1}B$$

Here we can choose

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 \cdots & 0 \\ & & \ddots \ 1 & \ddots & 0 \\ & 0 & & 0 & 1 \\ -d_0 & -d_1 & & -d_{\delta-2} & -d_{\delta-1} \end{bmatrix} x + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u \quad ; \quad y = [n_0, n_1, \cdots n_{\delta-1}]$$

Ex: $$T(s) = \frac{3s^2+5s+6}{s^3+2s^2+3s+4} \qquad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad \delta[T(s)] = \delta = 3$$

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -4 & -3 & -2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u \quad ; \quad y = [6 \ \ 5 \ \ 3]$$

$$T(s) = [6 \ \ 5 \ \ 3] \begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 4 & 3 & s+2 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \ ; \quad \det[s1_3 - A] = s(s(s+2)+3) + 4(1-0) = s^3+2s^2+3s+4$$

$$= \frac{[6 \ \ 5 \ \ 3]}{s^3+2s^2+3s+4} \begin{bmatrix} \text{last} \\ \text{column} \\ \text{of} \\ (s1_3-A)^{-1} \\ \times \det \end{bmatrix} ;$$

entries of last column

$(1,3) = (-1)^{3+1}\det\begin{bmatrix} -1 & 0 \\ s & -1 \end{bmatrix} = 1$

$(2,3) = (-1)^{2+3}\det\begin{bmatrix} s & 0 \\ 0 & -1 \end{bmatrix} = s$

$(3,3) = (-1)^{3+3}\det\begin{bmatrix} s & -1 \\ 0 & s \end{bmatrix} = s^2$

$$= [6 \ \ 5 \ \ 3] \begin{bmatrix} 1 \\ s \\ s^2 \end{bmatrix} \times \frac{1}{s^3+2s^2+3s+4}$$

$$= \frac{6+5s+3s^2}{s^3+2s^2+3s+4}$$

for general case the numerator will be

$$\underbrace{[n_0, n_1, \ldots, n_{\delta-1}]}_{c} \times \begin{bmatrix} \text{last column} \\ \text{of } (s1_\delta - A)^{-1} \\ \times \det(s1_\delta - A) \\ \\ \end{bmatrix} = c \begin{bmatrix} 1 \\ s \\ s^2 \\ \vdots \\ s^{\delta-1} \end{bmatrix} = n_0 + n_1 s + \cdots + n_{\delta-1} s^{\delta-1}$$

for the last column of $(s1_\delta - A)^{-1}$

$$\begin{bmatrix} s & -1 & 0 & \cdots & & 0 \\ 0 & s & -1 & 0 & \cdots & 0 \\ & & & & & 0 \\ & & & & s & -1 \\ d_0 & d_1 & \cdots & \cdots & d_{\delta-2} & s+d_{\delta-1} \end{bmatrix}$$

for the (i, δ) entry

$= (-1)^{i+\delta} M(i\text{th col}, \delta \text{ row})$

" minor

$$\det \begin{bmatrix} s & & & & \\ & s & & 0 & \\ & & s & & \\ 0 & & -1 & \ddots & \\ & & & & -1 \end{bmatrix}$$

$i-1 \rightarrow$ $\leftarrow i+1$

$\doteq \delta - 1$

$= (-1)^{i+\delta} s^{i-1} (-1)^{(\delta-1)-(i+1)} = s^{i-1} (-1)^{2\delta-2}$

$= s^{i-1}$

$\therefore$ this gives a minimal "realization"

$\dot{x} = Ax + Bu$

$y = cx$

of $T(s) = \left(\sum_{i=0}^{\delta-1} n_i s^i\right) \Big/ \left(s^\delta + \sum_{i=0}^{\delta-1} d_i s^i\right)$

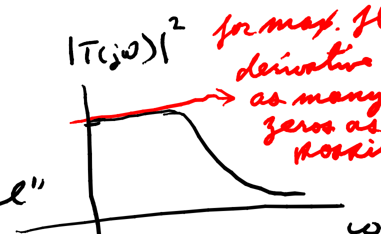
(here $T(s) = 0$ at $s = \infty$)

minimal

size of $x = \delta[T(s)] = \delta$

How to get transfer functions

Maximally flat, low pass



for max. flat derivative has as many zeros as possible

$T(s) \rightarrow |T(j\omega)|^2 = T(j\omega) T^*(j\omega)$

$= T(j\omega) T(-j\omega)$ if "real"

$\Rightarrow |T(j\omega)|^2$ is even in ω (i.e. $|T(j\omega)|^2 = |T(-j\omega)|^2$

$$|T(j\omega)|^2 = f(\omega^2) = \frac{1}{\omega^{2n} + a_{2n-2}\omega^{2n-2} + \cdots + a_2 \omega^2 + a_0}$$

$$f(x) = \frac{1}{x^n + a_{2n-2} x^{n-1} + \cdots + a_2 x + a_0} \quad ; \quad \frac{d f(\omega^2)}{d\omega^2} = \frac{d f(x)}{dx}$$

$=$

$$= \frac{1}{D(x)}$$

$$\frac{d f(x)}{dx} = -\frac{1}{D(x)^2} \cdot \frac{d D(x)}{dx} \Rightarrow$$

for maximally flat desire as many derivatives of $D(x)$ to be zero at $x = 0$

in a Taylor series expansion

$$= -\frac{1}{D(x)^2} \cdot \left[a_2 + 2a_3 x + \cdots (n-1) a_{2n-2} x^{n-2} + x^{n-1}\right]$$

set to zero to get zeros of Taylor series

$$f(x) = \frac{1}{x^n + a_0} \Rightarrow T(j\omega)\,T(-j\omega) = \frac{1}{\omega^{2n} + a_0}$$

can normalize $a_0 > 0$ to 1 by dividing $\frac{1/a_0}{(\omega/a_0^{-2n})^{2n}+1} \Rightarrow \frac{1}{\hat{\omega}^{2n}+1}$

$$T(j\omega)\cdot T(-j\omega) = \frac{1}{\omega^{2n}+1} \quad ; \; j = \sqrt{-1}$$

let $\omega \to s/j \Rightarrow T(s)\cdot T(-s) = \frac{1}{\left(\frac{s}{j}\right)^{2n}+1} = \frac{1}{s^n(-s)^n+1} = \frac{1}{(-1)^n s^{2n}+1}$

now have an analytic function in s, $T(s)T(-s)$ has $2n$ poles, the roots of $(-1)^n s^{2n} = -1 \Rightarrow s^{2n} = (-1)^{n+1}$

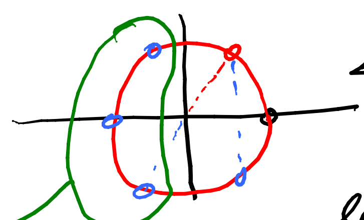
$\therefore$ desire the $2n$ roots of -1 or $+1$

case 1: n odd $\quad s^{2n} = 1 = e^{j2\pi k}$, $k = 0, 1, \dots$

$$\Rightarrow s_k = e^{j\frac{2\pi k}{2n}}, \quad k = 0, 1, \dots, 2n-1$$

$n = 3$

$\frac{360^\circ}{6 = 2n} = 60^\circ$



zeros of denominator of $T(s)T(-s)$

need for stability to choose left half plane zeros as in denominator

zeros of denominator of $T(s)$

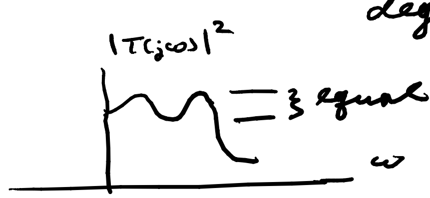
$$T(s) = \frac{1}{(s+1)(s+\underset{1/2}{\cos 60^\circ} + j\underset{\sqrt{3}/2}{\sin 60^\circ})(s+\cos 60^\circ - j\sin 60^\circ)}$$

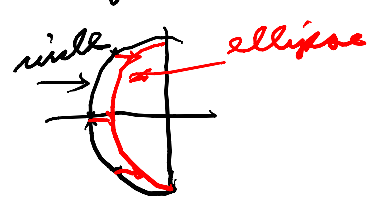
$$= \frac{1}{(s+1)(s+\frac{1}{2}+j\frac{\sqrt{3}}{2})(s+\frac{1}{2}-j\sqrt{3}/2)}$$

$$= \frac{1}{(s+1)(s^2+\frac{1}{4}+\frac{3}{4}+(\frac{1}{2}+\frac{1}{2})s)} = \frac{1}{(s+1)(s^2+s+1)}$$

$$= \frac{1}{s^3+2s^2+2s+1}$$ ⇐ universal low-pass maximally flat degree 3 "filter"

Equal ripple





for high pass set $s \to 1/s$

" band pass set $s \to \frac{s}{\omega_0} + \frac{\omega_0}{s}$