EE 610
10/13/09

°Testing:

Given $H = H^{T*}$ Hermitian

positive definite $x^{T*} H x > 0$ for all vectors $x \neq 0$

(semi) ≥ 0

for $H = \begin{bmatrix} h_1 & h_2 \\ h_2^* & h_3 \end{bmatrix}$ h_1 & h_3 real

for positive definite $h_1 > 0$ & $\det H = h_1 h_3 - |h_2|^2 > 0$

(if $\det H = 0$ & $h_2 \geq 0$ & $h_1 \geq 0$ is positive semidefinite)

$$[x_1^* \;\; x_2^*] \begin{bmatrix} h_1 & h_2 \\ h_2^* & h_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} > 0$$ for positive definiteness for all $x \neq 0$

Case 1: $x_2 = 0 \Rightarrow x^{T*} H x = x_1^* h_1 x_1 = |x_1|^2 h_1 > 0 \Rightarrow h_1 > 0$ for pos. def.

Case 2: $x_1 = 0 \Rightarrow$ $= x_2^* h_3 x_2 = |x_2|^2 h_2 \Rightarrow h_2 > 0$

Case 3: Find a transformation T to bring H to diagonal form $x = T\hat{x}$, $\det T \neq 0$

$$x^{T*} H x = \hat{x}^{T*} T^{T*} H T \hat{x} = \hat{x}^{T*} \hat{H} \hat{x}$$

choose $\hat{T}^{T*}$ to force $\hat{h}_{21} = 0$

$h_1^* = h_1$ as h_1 real

$$\begin{bmatrix} 1 & 0 \\ -\frac{h_2^*}{h_1} & 1 \end{bmatrix} \begin{bmatrix} h_1 & h_2 \\ h_2^* & h_3 \end{bmatrix} \begin{bmatrix} 1 & -h_2/h_1^* \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} h_1 & h_2 \\ 0 & h_3 - \frac{|h_1|^2}{h_1} \end{bmatrix} \begin{bmatrix} 1 & -h_2/h_1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} h_1 & 0 \\ 0 & \frac{h_3 h_1 - |h_2|^2}{h_1} \end{bmatrix}$$

$$x^{T*} H x = \hat{x}^{T*} \begin{bmatrix} h_1 & 0 \\ 0 & \frac{\det H}{h_1} \end{bmatrix} \hat{x},$$

where $h_1 > 0$ for pos. def.
as $\hat{x}$ is free to choose
$\Rightarrow \det H > 0$

Passive $\Rightarrow$ positive-real Z or Y
& bounded-real S

$A(s)$ is positive real (scalar)

if

1) $A(s)$ is real for $s = \sigma > 0$ (real circuit)

2) $A(s)$ is analytic in $s = \sigma + j\omega$, $\sigma > 0$ (stability)

3) $\mathrm{Re}\, A(s) \geqslant 0$ in $\sigma > 0$ (passivity)

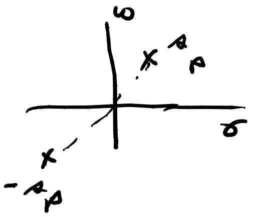
PR = rational positive real

1) $A(s)$ has real coefficients

2) $A(s)$ has no poles in $\sigma > 0$

3) $\mathrm{Re}\, A(s) \geqslant 0$ in $\sigma > 0$

4) $A(s) = -A(-s)$



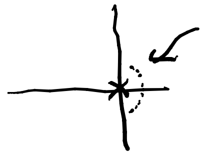
Lossless:

means no poles in $\sigma > 0$ or $\sigma < 0$

$\therefore$ all poles are on the $j\omega$ axis

Look at a possible pole at $s = 0$ near the pole the behavior is as $\dfrac{k}{s^m}$

for



take $s = re^{j\theta}$, $r > 0$, $-\frac{\pi}{2} < \theta < \pi/2$

$$A(s) \sim \frac{k}{s^m} = \frac{k}{r^m} e^{-jm\theta} = \frac{|k|}{r^m}\left(\cos(m\theta \overset{+\angle k}{\wedge}) + j\sin(m\theta \overset{\angle k}{\wedge})\right)$$

$$\mathrm{Re}\, A(s)\Big|_{\substack{\text{on this circle} \\ \text{in } \sigma > 0}} = \frac{|k|}{r^m} \cos(m\theta \overset{\angle k}{\wedge})$$

goes < 0 at some θ, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ if $m > 1$ & $\angle k = 0$

$$\sim \frac{k}{r} \cos\theta \quad > 0 \Rightarrow k > 0$$

This holds for other poles on $s = j\omega$

$\Rightarrow$ all poles of a lossless $y(s)$ or $z(s)$ are on the $j\omega$ axis, are simple, have real residues which are positive.

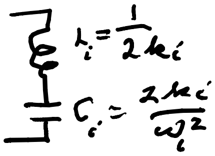
$\therefore$ the partial fraction expansion is

$$y(s) = \frac{k_0}{s} + k_\infty s + \sum_{i=1}^{N} \frac{k_1}{s + j\omega_1} + \frac{k_1}{s - j\omega_1} + \ldots \frac{k_i}{s + j\omega_i} + \frac{k_i}{s - j\omega_i}$$

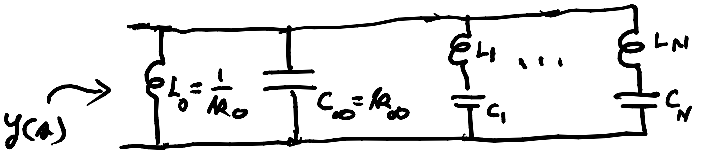
for a given pole pair $\dfrac{k_i}{s + j\omega_i} + \dfrac{k_i}{s - j\omega_i} = \dfrac{2k_i s}{s^2 + \omega_i^2} = y_i(s)$

$$= \frac{1}{\dfrac{s^2}{2k_i s} + \dfrac{\omega_i^2}{2k_i s}} = \frac{1}{z_i(s)}$$

synthesis of $z_i(s) = \frac{s}{2k_i} + \frac{1}{\left(\frac{2k_i}{\omega_i^2}\right)s}$



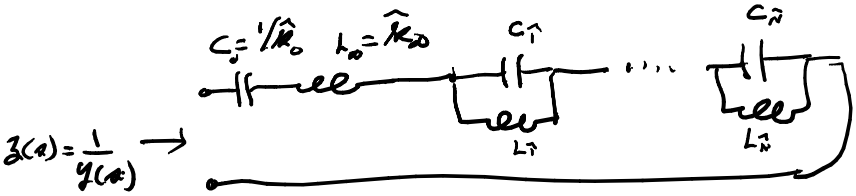
given a PR lossless $y(s)$



= 2nd Foster form

1st Foster do the same on $z(s)$

$$z(s) = \frac{\hat{k}_0}{s} + \hat{k}_\infty s + \sum_{i=1}^{\hat{N}} \frac{2\hat{k}_i s}{s^2 + \hat{\omega}_i^2}$$

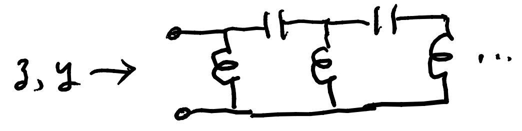


= 1st Foster

1st Cauer, remove poles at ∞ (continued fraction expansion about ∞)

(good for low pass filtering)

2nd Cauer, remove poles at 0 (" " about 0)



(good for high pass filtering)

Ex 1 $y(s) = \frac{s(s^2+6)}{(s^2+4)(s^2+9)}$; $y(-s) = \frac{-s(s^2+6)}{(s^2+4)(s^2+9)} = -y(s)$

partial fraction expansion

$$y(s) = \frac{k_1}{s+j2} + \frac{k_1^*}{s-j2} + \frac{k_2}{s+j3} + \frac{k_2^*}{s-j3}$$

$$k_1 : \ y(s)\times(s+j2)\Big|_{s=-j2} = \frac{s(s^2+6)}{(s-j2)(s^2+9)}\Big|_{s=-j2} = \frac{-j2(-4+6)}{-j4(-4+9)} = \frac{1}{2}\left(\frac{2}{5}\right) = 1/5$$

$$= \left[k_1 + \frac{k_1^*(s+j2)}{s-j2} + \frac{k_2(s+j2)}{s+j3} + \frac{k_2^*(s+j2)}{s-j3}\right]_{s=-j2}$$

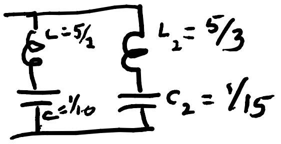
$$= k_1$$

$$\Rightarrow \quad k_1 = 1/5$$

$$k_2 = \left.\frac{s(s^2+6)}{(s^2+4)(s-j3)}\right|_{s=-j3} = \frac{-j3}{-j6}\cdot\frac{(-9+6)}{(-9+4)} = \frac{1}{2}\left(\frac{-3}{-5}\right) = \frac{3}{10}$$

$$y(s) = \frac{s(s^2+6)}{(s^2+4)(s^2+9)} = \frac{\frac{2}{5}s}{s^2+4} + \frac{\frac{3}{5}s}{s^2+9} \Rightarrow$$

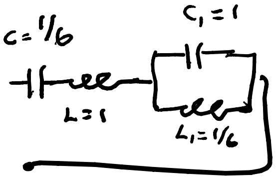
$$= \frac{1}{\frac{5}{2}s + \frac{10}{s}} + \frac{1}{\frac{5}{3}s + \frac{15}{s}}$$



2nd Foster

for 1st Foster

$$z(s) = \frac{1}{y(s)} = \frac{(s^2+4)(s^2+9)}{s(s^2+6)} = \frac{\frac{4\times9}{6}}{s} + 1\cdot s + \frac{1\,s}{s^2+6} \Rightarrow$$



1st Foster

$$\frac{2k_1 s}{s^2+6} \Rightarrow \frac{s^2+6}{s}\times z(s) = \left.\frac{\cancel{(s^2+6)}(s^2+4)(s^2+9)}{s^2\cancel{(s^2+6)}}\right|_{s^2=-6} = 2k_1$$

$$= \frac{(-6+4)(-6+9)}{-6} = \frac{-2}{-6}(3) = 1$$

1st Cauer: removes poles @ ∞

$$z(s) = \frac{(s^2+4)(s^2+9)}{s(s^2+6)} = \frac{s^4+13s^2+36}{s^3+6s} = s + \cfrac{1}{\frac{1}{7}s + \cfrac{1}{\frac{49}{8}s + \cfrac{1}{\frac{8}{7}\times 36 s}}}$$

$$s^3+6s \,\big)\, s^4+13s^2+36 \quad s$$

$$s^4+6s^2$$

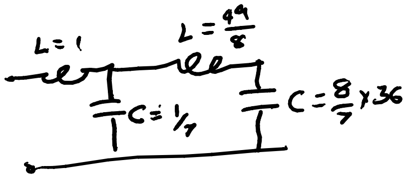
$$7s^2+36 \,\big)\, s^3+6s \quad \frac{1}{7}s$$

$$s^2+\frac{36}{7}s$$

$$\frac{8}{7}s \,\big)\, 7s^2+36 \quad \frac{49}{8}s$$

$$7s^2$$

$$36 \,\big)\, \frac{8}{7}s \quad \frac{8}{7}\times\frac{1}{36}s$$



1st Cauer