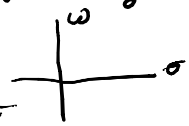
Phasors: EE 610

10/08/09

$e^{j\omega t} = \cos\omega t + j\sin\omega t, \quad j=\sqrt{-1}$

if linear can apply $\cos\omega t$, solve, & apply $\sin\omega t$ & solve or can apply $e^{j\omega t}$ & take Re or Im parts

But $e^{st} = e^{\sigma t + j\omega t}$ $\quad s = \sigma + j\omega \Rightarrow e^{j\omega t}$ is special case of e^{st}



$L(x) = F(u)$

$= F(e^{st}) = F(s)e^{st}$

if linear (time-invariant) $L(e^{st}) = L(s)e^{st}, \quad -\infty < t < \infty$

if $s = j\omega$; $L(s) = L(j\omega) = |L(j\omega)|e^{j\phi}$

If we apply $u(t) = V\cos(\omega t + \phi_u) = Re(|U|e^{j\phi_u}e^{j\omega t})$

can work with $|U|e^{j\phi_u}$ call this a phasor

Ex: $3\ddot{v}(t) + 2\dot{v} + 5v = 6\dot{u} + 2u$

if $u(t) = Ue^{st}$ $\quad = 6Use^{st} + 2Ue^{st} = (6s+2)Ue^{st}$

assume $v(t) = Ve^{st}$

on left $3Vs^2e^{st} + 2Vse^{st} + 5Ve^{st}$

$(3s^2+2s+5)Ve^{st} = (6s+2)Ue^{st}$; e^{st} is entire

$(3s^2+2s+5)V = (6s+2)U \quad \Rightarrow \frac{V}{U}(s) = \frac{3s^2+2s+5}{6s+2}$

if $s = j\omega$

$(-3\omega^2 + 2j\omega + 5)V = (6j\omega + 2)U$ $\quad a^* \Rightarrow j \to -j$

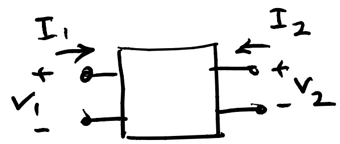
$([5-3\omega^2] + j[2\omega])V = (2 + j6\omega)U$

$$\frac{V}{U}(j\omega) = \frac{2+j6\omega}{[5-3\omega^2]+j2\omega} = \frac{(2+j6\omega)([5-3\omega^2]-j2\omega)}{([5-3\omega^2]+j2\omega)([5-3\omega^2]-j2\omega)}$$

$$= \frac{[2(5-3\omega^2) + 6\omega(2\omega)] + j[6\omega(5-3\omega^2) - 4\omega]}{(5-3\omega^2)^2 + (2\omega)^2}$$

$$Re\left(\frac{V}{U}(j\omega)\right) = \frac{10+6\omega^2}{(5-3\omega^2)^2+(2\omega)^2}$$

Given $Z(j\omega)$, for a 2-port this is 2×2 rational $Z(s)$ with real coefficients



$$v(\infty) = \int_{-\infty}^{\infty} v^T(t)\, i(t)\, dt = \int_{-\infty}^{\infty} \mathcal{V}^{*T}(\omega)\, \mathcal{I}(\omega)\, d\omega$$

Parseval's theorem

$$P_{in}(j\omega) = Re\left(V^{T*} I\right) = Re\left(V_1^* I_1 + V_2^* I_2\right)$$

given $a = a_r + j a_i$ a number, a_r & a_i real "real" "imaginary parts"

$a^* = a_r - j a_i$

$$2(a + a^*) = 2a_r \Rightarrow a_r = \frac{a + a^*}{2}$$

if a number $a_r^T = a_r$, also $(ab)^* = a^* b^*$

if matrices $(AB)^T = B^T A^T$

$$P_{in}(j\omega) = \frac{V^{T*} I + (V^{T*} I)^*}{2} = \frac{V^{T*} I + V^{T**} I^*}{2} = \frac{V^{T*} I + V^T I^*}{2}$$

$$= \left(\frac{V^{T*} I}{2}\right) + \left(\frac{V^T I^*}{2}\right)^T = \frac{V^{T*} I}{2} + \frac{I^{T*} (V^T)^T}{2}$$

$$= \frac{V^{T*} I + I^{T*} V}{2} = Re(I^{T*} V) = Re(V^{T*} I)$$

$$2P_{in}(j\omega) = (ZI)^{T*} I + I^{T*}(ZI) \qquad V = ZI$$

$$= I^{T*} Z^{T*} I + I^{T*} Z I$$

$$P_{in}(j\omega) = I^{T*}\left[\frac{Z^{T*}(j\omega) + Z(j\omega)}{2}\right] I \qquad \text{an Hermitian Form}$$

$$Z_H = \frac{Z^{T*} + Z}{2} = \text{Hermitian part}$$

if passive $P_{in}(j\omega) \geq 0 \Rightarrow Z_H(j\omega)$ is positive semi-definite

$\therefore$ for any I, complex phasor,

$$I^{T*} Z_H(j\omega) I \geq 0$$

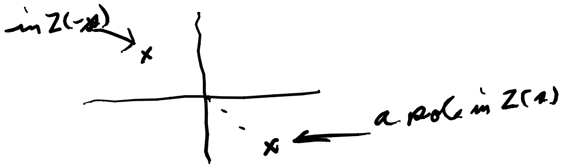
If lossless $Z_H(j\omega) = 0_{2\times2} \Rightarrow \dfrac{Z^{T*}(j\omega) + Z(j\omega)}{2} = 0_{2\times2}$

$$Z^T(-j\omega) = -Z(j\omega) \quad \text{if lossless (passive) rational with real coefficients}$$

lets replace ω by s/j to extend these rational functions into the $s = \sigma + j\omega$ plane if lossless: $Z^T(-j\cdot\frac{s}{j}) = -Z(j\frac{s}{j}) \Rightarrow Z^T(-s) = -Z(s)$

this shows no poles anywhere but on the $j\omega$ axis

$\frac{1}{-s+2} \Rightarrow \frac{-1}{s+2}$



if passive since a pole of $Z(s)$ in $\sigma > 0$ means instability (a small initial condition gives $e^{+\sigma t}$ ... cant be in a passive circuit

So a good way to check for passivity is to form $I^{T*} Z_H(j\omega) I$ for any complex vector I or $V^{T*} Y_H(j\omega) V$ for " " " V

or $\quad V^{i\,T*}\left[1_2 - S^{T*}(j\omega)S(j\omega)\right]V^i \geq 0$ for an complex incident vector V^i

if lossless then $1_2 = S^T(-s)S(s) \Rightarrow S^{-1}(s) = S^T(-s)$

also for a passive real rational $S(s)$ there are no poles on $s=j\omega$ axis (no poles in $\sigma>0$ by stability).

Back to semi-state

$$E\dot{x} = Ax + Bu$$
$$y = Cx$$

if not linear time invariant $Ax \to a(x,t)$

Take u = voltage input, y = currents output at the same ports

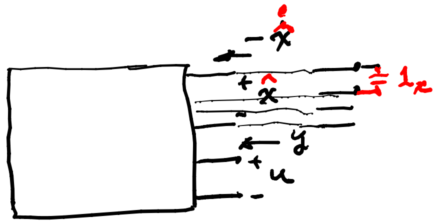
Find a P & a Q to make $E \Rightarrow \begin{bmatrix} 1_r & 0 \\ 0 & 0_{n-r}\end{bmatrix}$, $x = Q\hat{x}$

Q^{-1} exists & P^{-1} also & constants

$$PEQ\dot{\hat{x}} = PAQ\hat{x} + PBu$$
$$y = CQ\hat{x}$$

$PEQ = \begin{bmatrix} 1_r & 0 \\ 0 & 0\end{bmatrix}$; r = rank of E

will now this to design with r capacitors



$$-\begin{bmatrix}\dot{x}_n \\ 0\end{bmatrix} = -E\dot{\hat{x}} = \begin{bmatrix} \\ y\end{bmatrix} \qquad Y_{comp} = \underbrace{\begin{bmatrix}-PAQ & -PB \\ CQ & 0\end{bmatrix}}_{\text{can make with OTA's}} \begin{bmatrix}\hat{x} \\ u\end{bmatrix}$$