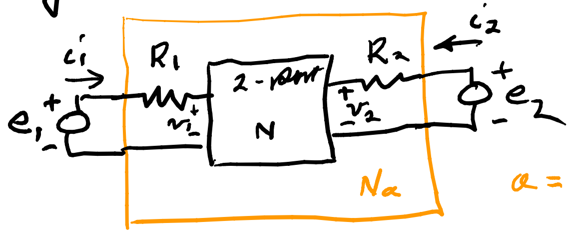
EE 610
09/24/09

# Scattering:




$a$ = augmented

By normalization can assume $R_1 = R_2 = 1$

$$2v^i = v + i$$
$$2v^r = v - i$$

can solve

add $2v^i + 2v^r = v + i + v - i = 2v$

subtract $2v^i - 2v^r = v + i - v + i = 2i$

$$\equiv \quad v = v^i + v^r$$
$$i = v^i - v^r$$

For linear N: $\quad v^r = S v^i \quad$ in frequency, $s$, domain

$$i = Y_a \cdot e \;; \quad e = i + v \Rightarrow v = e - i$$
$$v = e - Y_a e = (1_2 - Y_a) e$$

$$2v^i = v + i = e$$
$$2v^r = v - i = e - Y_a e - Y_a e = e - 2Y_a e = (1_2 - 2Y_a) e$$

$$\Rightarrow 2v^r = (1_2 - 2Y_a) \cdot 2v^i \quad \Rightarrow \quad S = 1_2 - 2Y_a$$

For the original 2-port N assume

$$Av = Bi \quad \Rightarrow \quad A(v^i + v^r) = B(v^i - v^r)$$
$$(A - B)v^i = (-B - A)v^r \Rightarrow \quad v^r = (B+A)^{-1}(B-A)v^i$$
$$\Rightarrow \quad S = (B+A)^{-1}(B-A)$$

$$(1_2 - Y_a) i = (1_2 - Y_a) Y_a \cdot e = (Y_a - Y_a^2) \cdot e$$

$$Y_a \cdot v = Y_a (1_2 - Y_a) e = (Y_a - Y_a^2) e$$

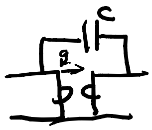
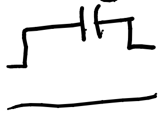
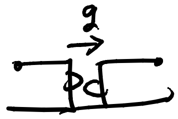
$$Av = Bi \quad \equiv \quad Y_a \cdot v = (1_2 - Y_a) i$$

$\therefore$ choose $A = Y_a$, $B = 1_2 - Y_a \quad \Rightarrow \quad B + A = Y_a + (1_2 - Y_a) = 1_2$

$$B - A = 1_2 - 2Y_a$$

If $N$ has an admittance, $i = Yv$ then can choose
$A = Y$, $B = 1_2 = \begin{bmatrix}1&0\\0&1\end{bmatrix}$ $\Downarrow$ $Av = Bi$

$$S = (B+A)^{-1}(B-A) = (1_2+Y)^{-1}(1_2-Y)$$

Ex:  =  +  $\Rightarrow Y = \begin{bmatrix}sC & -sC\\ -sC & sC\end{bmatrix} + \begin{bmatrix}0 & g\\ -g & 0\end{bmatrix}$

$$Y = \begin{bmatrix}sC & -sC+g\\ -sC-g & sC\end{bmatrix} \qquad 1_2 - Y = \begin{bmatrix}1-sC & sC-g\\ sC+g & 1-sC\end{bmatrix}$$

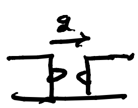
$$1_2+Y = \begin{bmatrix}1+sC & -sC+g\\ -sC-g & 1+sC\end{bmatrix}; \quad \det = (1+sC)^2 - (-sC-g)(-sC+g)$$
$$= 1+2sC+(sC)^2 - [(sC)^2 - g^2]$$
$$= (1+g^2+2sC)$$

$$(1_2+Y)^{-1} = \frac{1}{1+g^2+2sC}\begin{bmatrix}1+sC & sC-g\\ sC+g & 1+sC\end{bmatrix}$$

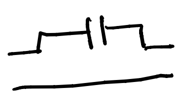
$$\times\begin{bmatrix}1-sC & sC-g\\ sC+g & 1-sC\end{bmatrix}$$

$$S = \frac{1}{1+g^2+2sC}\begin{bmatrix}1-\cancel{(sC)^2}+\cancel{(sC)^2}-g^2 & sC-g+\cancel{(sC)^2}-\cancel{sCg} + sC-\cancel{(sC)^2}-g+\cancel{sCg}\\ sC-\cancel{(sC)^2}+g-\cancel{sCg} + sC+g+\cancel{(sC)^2}+\cancel{sCg} & \cancel{(sC)^2}-g^2+1-\cancel{(sC)^2}\end{bmatrix},$$

$$S = \frac{1}{1+g^2+2sC}\begin{bmatrix}1-g^2 & -2g+2sC\\ 2g+2sC & 1-g^2\end{bmatrix}$$

for the gyrator itself  $S = \frac{1}{1+g^2}\begin{bmatrix}1-g^2 & -2g\\ 2g & 1-g^2\end{bmatrix}$ we set $C=0$

$$s_{11} = \left.\frac{v_1^r}{v_1^i}\right|_{v_2^i=0} = \text{input reflection coefficient}$$

for the capacitor itself  $S = \begin{bmatrix}\frac{1}{1+2sC} & \frac{2sC}{1+2sC}\\ \frac{2sC}{1+2sC} & \frac{1}{1+2sC}\end{bmatrix}$
$g=0$

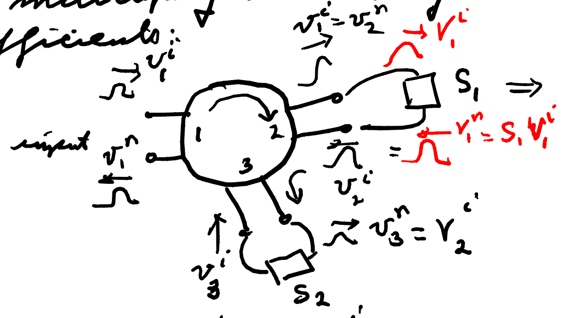
$Y = \begin{bmatrix}sC & -sC\\ -sC & sC\end{bmatrix}$ $= S^T$

$= Y^T$ & nonsingular

singular

$$\det = \frac{1-(2sC)^2}{(1+2sC)^2} = \frac{1-2sC}{1+2sC}$$

To multiply scattering matrices, size 1×1 = reflection coefficients:




$\Rightarrow \quad V_1^r = S_1 V_1^i \;, \quad V_1^i = v_2^r = v_1^i$

$v_3^i \;''\; V_2^r = S_2 V_2^i$

S for the loaded 3-port circulator

$$v_1^r = S v_1^i$$

$$v_1^i = V_1^i \to v_3^r = V_2^i$$

$$v_3^i = v_1^r = V_2^r = S_2 V_2^i$$

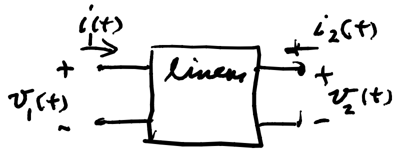
$$v_1^r = v_3^i = S_2 V_2^i = S_2 v_3^r$$
$$= S_2 v_2^i = S_2 V_1^r = S_2 S_1 V_1^i = S_2 \cdot S_1 v_1^i$$

$$S = S_2 \cdot S_1$$

so $S_{circ}$: $\begin{bmatrix} v_1^r \\ v_2^r \\ v_3^r \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} v_1^i \\ v_2^i \\ v_3^i \end{bmatrix}$ ; $S_{circ} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

Passivity: a circuit is passive if the total energy in is always positive. $\left.\begin{array}{l}\mathcal{E}(t) = \text{energy} \\ p(t) = \text{power}\end{array}\right\}$ at time $t$

$$\mathcal{E}(t) = \int_{-\infty}^{t} p(\tau)\,d\tau + \underbrace{\mathcal{E}(-\infty)}_{\text{assume } = 0}$$




$$p(t) = v_1(t) i_1(t) + v_2(t) i_2(t)$$
$$= \text{total power in}$$
$$= v^T(t)\, i(t)$$

$$\mathcal{E}(t) = \int_{-\infty}^{t} v^T(\tau)\, i(\tau)\, d\tau \geq 0 \text{ if passive}$$

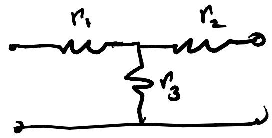
Ex: $i_1(t) = g\, v_2(t) \,, \quad i_2(t) = -g\, v_1(t)$ gyrator

$$p(t) = v_1 \cdot i_1 + v_2 i_2 = v_1 \cdot g \cdot v_2 + v_2(-g \cdot v_1) \equiv 0$$

$\equiv 0$ we call instantaneously lossless

$\Rightarrow \mathcal{E}(t) \geq 0$ & the gyrator is passive

Eg:




$\Rightarrow Z = \begin{bmatrix} r_1+r_3 & r_3 \\ r_3 & r_2+r_3 \end{bmatrix}$; $v = Z i$

$$p_{in}(t) = v_1 i_1 + v_2 i_2 = [(r_1+r_3) i_1 + r_3 i_2] i_1 + [r_3 i_1 + (r_2+r_3) i_2] i_2$$
$$= (r_1+r_3) i_1^2 + 2 r_3 i_1 i_2 + (r_2+r_3) i_2^2$$
$$= v^T i = (Z i)^T \cdot i = i^T Z^T i$$ quadratic form

this is a passive 2-port if this Z is positive semi-definite.