EE 610 M 10/13/08

Richards' functions

$$\hat{R}(s) = \frac{k f(k) - s f(s)}{k f(s) - s f(k)} \quad \text{is PR if } f(s) \text{ is PR \& } k>0$$

$$\text{if } s = k \Rightarrow \frac{k f(k) - k f(k)}{k f(k) - k f(k)} = \frac{0}{0}$$

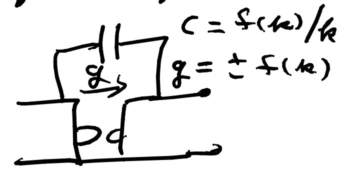
$\Rightarrow s-k$ factors if $f(s) \not\equiv 0$

if also $f(-k) = -f(k)$ then at $s=-k$

$$\frac{k f(k) - (-k) f(-k)}{k f(-k) - (-k) f(k)} = \frac{k f(k) - k f(k)}{-k f(k) + k f(k)} = \frac{0}{0}$$

in which case $s+k$ is a factor if $f(s) \not\equiv 0$

$$\frac{y_L(s)}{y(k)} = \hat{R}(s)\Big|_{f(s)=y(s)} = \frac{k\, y(k) - s\, y(s)}{k\, y(s) - s\, y(k)} \Longrightarrow$$



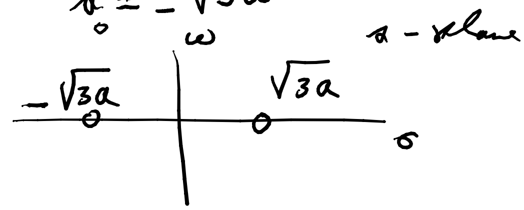
Ex: $y(s) = \dfrac{s+a}{s+3} \qquad a>0$

$$\left(y(s)+y(-s)\right) = 2\,Ev\, y(s) = \frac{s+a}{s+3} + \frac{-s+a}{-s+3} = \frac{(s+a)(-s+3) + (s+3)(-s+a)}{(s+3)(-s+3)}$$

$$= \frac{(-s^2+\cancel{3s}-\cancel{as}+3a) + (-s^2+\cancel{as}-\cancel{3s}+3a)}{-s^2+9}$$

$$Ev\, y(s) = \frac{-s^2+3a}{-s^2+9} \quad \text{if } a \neq 3 \Rightarrow s^2 - 3a = 0$$

$$s_o = \pm\sqrt{3a}$$



Choose for k, $k = +\sqrt{3a}$

we know $s^2 - 3a$ cancels in the Richards' function

$$\hat{R}(s) = \frac{k f(k) - s f(s)}{k f(s) - s f(k)} = \frac{\sqrt{3a}\cdot f(\sqrt{3a}) - s f(s)}{\sqrt{3a}\, f(s) - s f(\sqrt{3a})}$$

where $f(s) = y(s) = \dfrac{s+a}{s+3}$; $\quad f(\sqrt{3a}) = \dfrac{\sqrt{3a}+a}{\sqrt{3a}+3}$

$$\hat{R}(s) = \frac{\sqrt{3a}\cdot\left(\frac{\sqrt{3a}+a}{\sqrt{3a}+3}\right) - s\left(\frac{s+a}{s+3}\right)}{\sqrt{3a}\left(\frac{s+a}{s+3}\right) - s\left(\frac{\sqrt{3a}+a}{\sqrt{3a}+3}\right)} = \frac{-s^2 - as + (s+3)\left(\sqrt{3a}\left(\frac{\sqrt{3a}+a}{\sqrt{3a}+3}\right)\right)}{-(\sqrt{3a}+a)s^2 - s\left(\sqrt{3a}(\sqrt{3a}+3)\right)}$$

$$\frac{\sqrt{3a}\,s(\sqrt{3a}+3) + \sqrt{3a}\cdot a \times}{(\sqrt{3a}+3)}$$

$$(\sqrt{3a}+3)$$

$$= \frac{-s^2 + \frac{(s+3)(3a+a\sqrt{3a})}{\sqrt{3a}+3} - as}{\quad} \quad \times\frac{1}{s+3}$$

$$= \frac{1}{(\sqrt{3a}+3)\times(s+3)}\left(\frac{-s^2(\sqrt{3a}+3) + s\{\cancel{3a} + \cancel{a\sqrt{3a}} - \cancel{a\sqrt{3a}} - \cancel{3a}\} + 3(3a+a\sqrt{3a})}{\frac{-(\sqrt{3a}+a)s^2 + 0\cdot s + (3a^2+3a\sqrt{3})}{(s+3)(\sqrt{3a}+3)}}\right)$$

in detail

$$\sqrt{3a}\left(\frac{s+a}{s+3}\right) - s\left(\frac{\sqrt{3a}+a}{\sqrt{3a}+3}\right) = \frac{\sqrt{3a}(s+a)(\sqrt{3a}+3) - s(s+3)(\sqrt{3a}+a)}{(s+3)(\sqrt{3a}+3)}$$

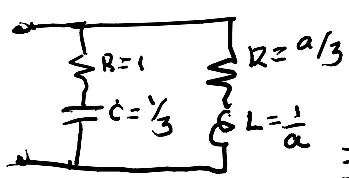
$$= \frac{-s^2(\sqrt{3a}+a) + s\left(\sqrt{3a}(\sqrt{3a}+\cancel{3}) - 3(\cancel{\sqrt{3a}}+a)\right) + \sqrt{3a}\cdot a(\sqrt{3a}+3)}{(s+3)(\sqrt{3a}+3)}$$

$$\hat{R}(s) = \frac{-s^2(\sqrt{3a}+3) + 3a(3+\sqrt{3a})}{-s^2(\sqrt{3a}+a) + 3a^2+3a\sqrt{3a}} = \left(\frac{\sqrt{3a}+3}{\sqrt{3a}+a}\right)\cdot\frac{s^2 - \frac{3a(3+\sqrt{3a})}{\sqrt{3a}+3}}{s^2 - \frac{3a(a+\sqrt{3a})}{\sqrt{3a}+a}} = \left(\frac{\sqrt{3a}+3}{\sqrt{3a}+a}\right)$$

is $s^2 = \frac{3a(3+\sqrt{3a})}{\sqrt{3a}+3} = \frac{3a(a+\sqrt{3a})}{\sqrt{3a}+a} = 3a$

if not need to back & check as $\hat{R}(s) = \text{const} = \left(\frac{\sqrt{3a}+3}{\sqrt{3a}+a}\right)$

$$y(s) = \frac{s+a}{s+3} = \frac{s}{s+3} + \frac{a}{s+3} = \frac{1}{1+3/s} + \frac{1}{\frac{s}{a}+3/a}$$

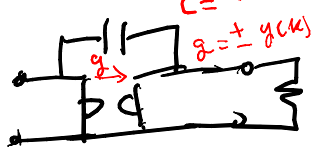


$\equiv$

uses 2 resistors

$C = y(s)/s = \frac{\sqrt{3a}+a}{\sqrt{3a}(\sqrt{3a}+3)}$

$g = \frac{1}{s}y(s) = \frac{\sqrt{3a}+a}{\sqrt{3a}+3}$



$R \Rightarrow \frac{1}{R} = G = y(s)\,\hat{R}(s) = 1$

Note: The Richards' function still works if $a=3$ when $y(s)=1$; $C=1/3$, $g=1$ & load $y_L=1$

Reason works: $y_2(s)$ is PR

Look at scattering; use $2v^i = v + z_0 i$ normalize

$2v^r = v - z_0 i$ $\quad z_0 = 1$

$$v^r = S \cdot v^i \Rightarrow 2v^r = S \cdot 2v^i$$

$$v - z_0 i = S(v + z_0 i)$$

Let $v = Z \cdot i \Rightarrow Z \cdot i - z_0 \cdot i = S(Z \cdot i + z_0 \cdot i)$

$$(Z - z_0) \cdot i = S(Z + z_0) i$$

$$S = (Z - z_0)(Z + z_0)^{-1}$$

if $i = Y \cdot v$ $\quad 2v^r = v - z_0 i = (1 - z_0 Y) v$

$$= S(2v^i) = S(v + z_0 i)$$

$$= S(v + z_0 Y v)$$

$$= S(1 + z_0 Y) v$$

$$S = (1 - z_0 Y)(1 + z_0 Y)^{-1} = (y_0 - Y)(y_0 + Y)^{-1}$$

Normalize $y_0 = 1 \Rightarrow S = (1 - Y)(1 + Y)^{-1}$

$$= (Z - 1)(1 + Z)^{-1}$$

If $Z \to Y$ i.e. $Z_d = Y_d$ then $S_d = -S$ $\therefore$ $S < 0$ is possible so S is not PR if Y or Z are

Consider a 1-port, $Y(s) = y(s)$

$y(s)$ is PR look@ $S = \dfrac{(1 - y)}{(1 + y)}$ in $\sigma > 0$

$y(s) = g(s) + j b(s)$; $g = \mathrm{Re}\, y$, $b = \mathrm{Im}\, y = \dfrac{y(s) - y^*(s)}{2j}$

$= \dfrac{y(s) + y^*(s)}{2}$

we know

$g(s) \geq 0$ in $\sigma > 0$ for $y(s)$ PR

$$S = \frac{1 - g - jb}{1 + g + jb} \Rightarrow |S(s)|^2 = \frac{((1-g)^2 + b^2)}{((1+g)^2 + b^2)}$$

but $1 - g < 1 + g$ as $g > 0$ for y PR

$\therefore \quad |S(s)|^2 \leq 1$ in $\sigma > 0$ if $y(s)$ is PR

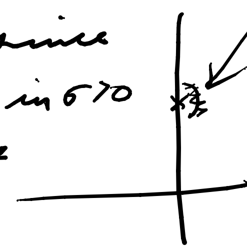
$\therefore \quad 1 - S(s)S^*(s) \geq 0$ in $\sigma > 0$ if the circuit is passive

This is part of the bounded-real condition

$S(s)$ is bounded real by definition if

1) $S(s)$ is real for real s in $\sigma > 0$
2) $S(s)$ is analytic in $\sigma > 0$
3) $1 - S(s)S^*(s) \geq 0$ in $\sigma > 0$

$S(s)$ has no poles on the $j\omega$ axis since otherwise $|S| \to \infty$ in $\sigma > 0$ near the pole



$|S|$ would be very large near a pole

Look at $S(s)$ for $y_L(s)$:

$$S = \frac{1-y_L}{1+y_L} = \frac{1 - \dfrac{k f(k) - s f(s)}{k f(s) - s f(k)}}{1 + \dfrac{k f(k) - s f(s)}{k f(s) - s f(k)}} = \frac{k y(s) - s y(k) - k y(k) + s y(s)}{k y(s) - s y(k) + k y(k) - s y(s)}$$

$$= \frac{k+s}{k-s} \cdot \frac{(-y(k) + y(s))}{(y(k) + y(s))} = \frac{1 + s/k}{1 - s/k} \cdot \frac{1 - \dfrac{y(s)}{y(k)}}{1 + \dfrac{y(s)}{y(k)}} \times (-1)$$