Semi-state equations
in time-domain, linear time-invariant

$$E\frac{dx}{dt} = Ax + Bu \qquad x = \text{semi-state}$$

$$y = Cx$$

(can be $x = \begin{bmatrix} v_t \\ \vdots \\ i_g \end{bmatrix}$)

$u(t)$ = input
$y(t)$ = output

If take Laplace transform

$$E s X(s) = A X(s) + B U(s)$$

$$Y(s) = C X(s)$$

Solve for $X(s)$: $(Es - A)X(s) = BU(s)$

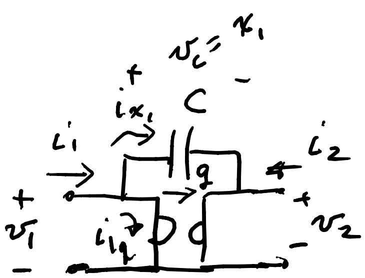
$$X(s) = (Es-A)^{-1} B \cdot U(s)$$

$$Y(s) = C\left[(Es-A)^{-1} B\right] U(s) \ ; \quad T(s) = \text{transfer function}$$

$$T(s) = C\left[(Es-A)^{-1} B\right]$$

note finite poles are in denominator of T or they zeros of $\det(Es - A) \Rightarrow$ characteristic equation is $\det(Es-A) = 0$

Ex:



desire $T(s) = Y(s)$ admittance

$$I = YV$$

$$u = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}, \quad y = \begin{bmatrix} i_1 \\ i_2 \end{bmatrix}$$

$C\frac{dx_1}{dt}$ = input i_1 + (− input current of gyrator 1st port
=

$$Y_{ad}(s) = \begin{bmatrix} sC & -sC+g \\ -sC-g & sC \end{bmatrix} \Rightarrow \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} C & -C \\ -C & C \end{bmatrix} d\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + \begin{bmatrix} 0 & g \\ -g & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

Choose $x_a = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$;

$$\begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = C\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\frac{dx}{dt} + \begin{bmatrix} 0 & g \\ -g & 0 \end{bmatrix} u$$

$$x_b = \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} \Rightarrow \qquad x_b = C\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\frac{dx_a}{dt} + \begin{bmatrix} 0 & g \\ -g & 0 \end{bmatrix} u$$

$$x_a = u$$

$$\begin{bmatrix} C & -C \\ -C & C \end{bmatrix}\frac{dx_a}{dt} = x_b - \begin{bmatrix} 0 & g \\ -g & 0 \end{bmatrix} u$$

$\Rightarrow$ has trouble as an s in the numerator of the transfer function.

another example to give s in the numerator

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\frac{dx}{dt} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} x + \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} u$$

$$y = [\gamma_1 \ \ \gamma_2]\, x$$

$$T(s) = [\gamma_1 \ \ \gamma_2]\left\{\begin{bmatrix} s & 1 \\ -1 & 0 \end{bmatrix}\right\}^{-1}\begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = [\gamma_1 \ \ \gamma_2]\,\frac{\begin{bmatrix} 0 & -1 \\ 1 & s \end{bmatrix}}{1}\begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}$$

$$= [\gamma_2 \quad -\gamma_1 + s\gamma_2]\begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \alpha_1\gamma_2 - \alpha_2\gamma_1 + s\gamma_2\alpha_2$$

$$= (\gamma_2\alpha_2)\, s \qquad\qquad \text{if } \alpha_1\gamma_2 = \alpha_2\gamma_1$$

For a 2×2 $T(s)$ with a pole of coefficient matrix rank 1

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\frac{dx}{dt} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} x + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} u$$

$$y = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} x$$

$$\begin{bmatrix} sc & -as+g \\ -as-g & +sc \end{bmatrix} = T(s) = \underbrace{\begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}}_{C}\underbrace{\begin{bmatrix} 0 & -1 \\ 1 & s \end{bmatrix}}_{(sE-A)^{-1}}\underbrace{\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}}_{B} = \overbrace{\begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & s \end{bmatrix}\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}}^{s\text{-dependent}}$$

$$= C(sE-A)^{-1}B \qquad + \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}\begin{bmatrix} 0 & -1 \\ +1 & 0 \end{bmatrix}\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

$$= s\begin{bmatrix}0 & c_{12}\\ 0 & c_{22}\end{bmatrix}\begin{bmatrix}b_{11} & b_{12}\\ b_{21} & b_{22}\end{bmatrix} + \begin{bmatrix}c_{12} & -c_{11}\\ +c_{22} & -c_{21}\end{bmatrix}\begin{bmatrix}b_{11} & b_{12}\\ b_{21} & b_{22}\end{bmatrix}$$

$$s\begin{bmatrix}c & -c\\ -c & c\end{bmatrix} + \begin{bmatrix}0 & g\\ -g & 0\end{bmatrix} = s\begin{bmatrix}\underbrace{c_{12}b_{21}}_{=c} & \underbrace{c_{12}b_{22}}_{=-c}\\ \underbrace{c_{22}b_{21}}_{=-c} & \underbrace{c_{22}b_{22}}_{=c}\end{bmatrix} + \begin{bmatrix}\underbrace{c_{12}b_{11}-c_{11}b_{21}}_{=0} & \underbrace{c_{12}b_{12}-c_{11}b_{22}}_{=g}\\ \underbrace{c_{22}b_{11}-c_{21}b_{21}}_{=-g} & \underbrace{c_{22}b_{12}-c_{21}b_{22}}_{=0}\end{bmatrix}$$

for the C_{ap} part

$c = c_{12}b_{21} = C_{ap} = c_{22}b_{22} = -c_{22}b_{21} = -c_{12}b_{22}$ for $c_{22} \neq 0$

$\Rightarrow b_{22} = -b_{21}$

$\Rightarrow c_{22} = -c_{12}$

for the gyrator part

(1,1) term $= c_{12}b_{11} - c_{11}b_{21} = 0 = c_{22}b_{12} - c_{21}b_{22}$ = (2,2) term

$\overset{\shortparallel}{-c_{22}}$ $\quad \overset{\shortparallel}{-b_{22}}$ $\quad = -c_{12}b_{12} - c_{21}(-b_{21}) \Rightarrow b_{12} = \dfrac{c_{21}}{c_{12}}b_{21}$

$= -c_{22}b_{11} + c_{11}b_{22} = 0 \quad \Rightarrow b_{11} = \dfrac{c_{11}}{c_{22}}b_{22} = \dfrac{c_{11}}{c_{12}}b_{21}$

desire the off diag to give $+g, -g$

$\Rightarrow c_{12}b_{12} - c_{11}b_{22} = g$ = (1,2) term $\Rightarrow c_{12}\left(\dfrac{c_{21}}{c_{12}}b_{21}\right) - c_{11}(-b_{21}) = g$

$c_{22}b_{11} - c_{21}b_{21} = -g$ = (2,1) term $\quad \Rightarrow b_{21} = g/(c_{21}+c_{11})$

continue off-line: $\quad \Rightarrow -g = \left(-c_{12}\cdot\dfrac{c_{11}}{c_{12}}b_{21}\right) - c_{21}b_{21}$

$= (-c_{11} - c_{21})b_{21}$ agrees so (2,1) = −(1,2) for "gyrator" part

Summary: $c = C_{capacitor}$

$c_{12} = \dfrac{c}{b_{21}}$, $c_{22} = \dfrac{c}{b_{22}} = -\dfrac{c}{b_{21}} = -c_{12}$, $b_{22} = \dfrac{-c}{c_{12}} = -b_{21}$, $b_{12} = \dfrac{c_{21}}{c_{12}}b_{21} = c_{21}\cdot\dfrac{b_{21}^2}{c}$

$b_{11} = \dfrac{c_{11}}{c_{12}}b_{21}$

$b_{21} = \dfrac{g}{c_{11}+c_{21}}$ with c_{11} & c_{21} free to be chosen, choose $c_{11} = c_{21} = \frac{1}{2}$

Then $b_{21} = g$, $c_{11} = 1/2$, $c_{12} = c/g$, $c_{21} = 1/2$, $c_{22} = -c/g$

$b_{11} = \dfrac{g^2}{2c}$, $b_{12} = \dfrac{g^2}{2c}$, $b_{22} = -g$

$$\therefore \underbrace{\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}}_{=E}\frac{dx}{dt} = \underbrace{\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}}_{=A}x + \underbrace{\begin{bmatrix}g^2/2c & g^2/2c\\ g & -g\end{bmatrix}}_{=B}\underbrace{\begin{bmatrix}v_1\\ v_2\end{bmatrix}}_{=u}$$

$$y = \begin{bmatrix}i_1\\ i_2\end{bmatrix} = \underbrace{\begin{bmatrix}1/2 & c/g\\ 1/2 & -c/g\end{bmatrix}}_{=C}x$$

should give $Y_{2\text{-port}} = \begin{bmatrix} sc & -sc+g \\ -sc-g & sc \end{bmatrix} = C(Es-A)^{-1}B$

as a check:

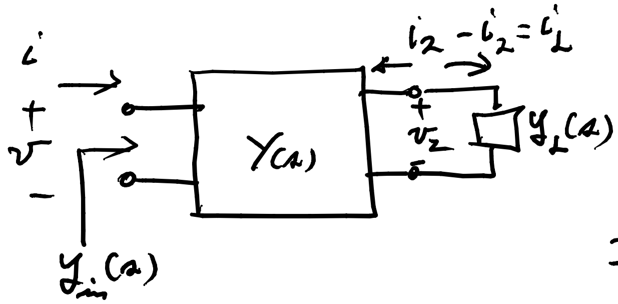
$$C(Es-A)^{-1}B = \begin{bmatrix} 1/2 & c/g \\ 1/2 & -c/g \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & s \end{bmatrix} \begin{bmatrix} g^2/2c & g^2/2c \\ g & -g \end{bmatrix}$$

$$= \begin{bmatrix} c/g & -\frac{1}{2} + \frac{sc}{g} \\ -c/g & -\frac{1}{2} - \frac{sc}{g} \end{bmatrix} \begin{bmatrix} g^2/2c & g^2/2c \\ g & -g \end{bmatrix}$$

$$= \begin{bmatrix} \frac{c}{g}\cdot\frac{g^2}{2c} - \frac{1}{2}g + \frac{sc}{g}\cdot g & \frac{c}{g}\cdot\frac{g^2}{2c} + \frac{g}{2} + \frac{sc}{g}(-g) \\ -\frac{c}{g}\cdot\frac{g^2}{2c} - \frac{1}{2}g - \frac{sc}{g}\cdot g & -\frac{c}{g}\cdot\frac{g^2}{2c} + \frac{g}{2} - \frac{sc}{g}(-g) \end{bmatrix}$$

$$= \begin{bmatrix} sc & g-sc \\ -g-sc & sc \end{bmatrix}$$ which checks

end off line



$$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

$$I_2 = -I_L = -y_L(s)\cdot V_2$$

$$\begin{bmatrix} I_1 \\ -y_L \cdot V_2 \end{bmatrix} = \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \Rightarrow \begin{bmatrix} I_1 \\ 0 \end{bmatrix} = \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22}+y_L \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

eliminate V_2: $\quad -y_{21}V_1 = (y_{22}+y_L)V_2$

if $y_{22}+y_L$ is nonsingular

$$V_2 = -(y_{22}+y_L)^{-1} y_{21} V_1$$

$$I_1 = y_{11}V_1 - y_{12}(y_{22}+y_L)^{-1} y_{21} V_1$$

$$y_{in}(s) = y_{11}(s) - y_{12}(y_{22}+y_L)^{-1} y_{21} = \frac{\Delta Y + y_{11}y_L}{y_{22}+y_L}$$

Ex: $Y_{2\text{-port}} = \begin{bmatrix} sC & -sC+g \\ -sC-g & sC \end{bmatrix}$, $\Delta Y_{2\text{-port}} = g^2$

$$y_{in} = sC - (-sC+g)(sC+y_L)^{-1}(-sC-g)$$

$$= \frac{sC(sC+y_L) + (-sC+g)(sC+g)}{sC+y_L} = \frac{sCy_L + g^2}{sC+y_L}$$

given y_{in} find y_L: $(sC+y_L)y_{in} = sCy_L + g^2$

$$sCy_{in} + y_L y_{in} = sCy_L + g^2$$

$$(y_{in} - sC)y_L = g^2 - sCy_{in}$$

$$y_L(s) = \frac{g^2 - sCy_{in}(s)}{y_{in}(s) - sC}$$

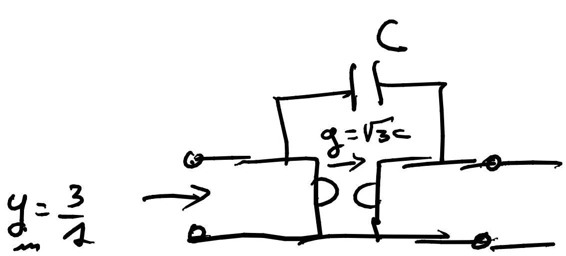
can use this for synthesis if can find a g & a C to give y_L simpler than y_{in}

Ex: $y_{in}(s) = 3/s$ form $y_L(s) = \dfrac{g^2 - sC\cdot 3/s}{\frac{3}{s} - sC}$

$$= \frac{s(g^2 - 3C)}{3 - s^2C}$$

choose $g^2 = 3C \Rightarrow y_L(s) = 0$

an open



Next will be positive-real functions & their synthesis.