Richards' Function for Cascade

10/07/07

$$y_R = y(k_1)\left[\frac{k_1 y(k_1) - s\,y(s)}{k_1 y(s) - s\,y(k_1)}\right]$$

$$y_{RR} = y_R(k_2)\left[\frac{k_2 y_R(k_2) - s\,y_R(s)}{k_2 y_R(s) - s\,y_R(k_2)}\right]$$

$$= y(k_1)\left[\frac{k_1 y(k_1) - k_2 y(k_2)}{k_1 y(k_2) - k_2 y(k_2)}\right]\left[\frac{k_2\, y(k_1)\left[\dfrac{k_1 y(k_1) - k_2 y(k_2)}{k_1 y(k_2) - k_2 y(k_1)}\right] - s\,y(k_1)\left[\dfrac{k_1 y(k_1) - s\,y(s)}{k_1 y(s) - s\,y(k_1)}\right]}{k_2\, y(k_1)\left[\dfrac{k_1 y(k_1) - s\,y(s)}{k_1 y(s) - s\,y(k_1)}\right] - s\,y(k_1)\left[\dfrac{k_1 y(k_1) - k_2 y(k_2)}{k_1 y(k_2) - k_2 y(k_1)}\right]}\right]$$

$$= y(k_1)\left[\frac{k_1 y(k_1) - k_2 y(k_2)}{k_1 y(k_2) - k_2 y(k_1)}\right]\left[\frac{k_2[k_1 y(k_1) - k_2 y(k_2)][k_1 y(s) - s\,y(k_1)] - s[k_1 y(k_1) - s\,y(s)][k_1 y(k_2) - k_2 y(k_1)]}{k_2[k_1 y(k_1) - s\,y(s)][k_1 y(k_2) - k_2 y(k_1)] - s[k_1 y(k_1) - k_2 y(k_2)][k_1 y(s) - s\,y(k_1)]}\right]$$

$$= y(k_1)\left[\frac{k_1 y(k_1) - k_2 y(k_2)}{k_1 y(k_2) - k_2 y(k_1)}\right]\left[\frac{y(s)\{k_1 k_2[k_1 y(k_1) - k_2 y(k_2)] + s^2[k_1 y(k_2) - k_2 y(k_1)]\} - s\{k_2 y(k_1)[k_1 y(k_1) - k_2 y(k_2)] + k_1 y(k_1)[k_1 y(k_2) - k_2 y(k_1)]\}}{\{k_2 k_1 y(k_1)[k_1 y(k_2) - k_2 y(k_1)] + s^2 y(k_1)[k_1 y(k_1) - k_2 y(k_2)]\} - s\,y(s)\{k_2[k_1 y(k_2) - k_2 y(k_1)] + k_1[k_1 y(k_1) - k_2 y(k_2)]\}}\right]$$

$$= y(k_1)\left[\frac{k_1 y(k_1) - k_2 y(k_2)}{k_1 y(k_2) - k_2 y(k_1)}\right]^2 \frac{k_1 k_2}{k_2 k_1 y(k_1)}\left[\frac{y(s)\left\{1 + \dfrac{s^2}{k_1 k_2}\left[\dfrac{k_1 y(k_2) - k_2 y(k_1)}{k_1 y(k_1) - k_2 y(k_2)}\right]\right\} - s\left\{\dfrac{y(k_1)}{k_1} + \dfrac{y(k_1)}{k_2}\left[\dfrac{k_1 y(k_2) - k_2 y(k_1)}{k_1 y(k_1) - k_2 y(k_2)}\right]\right\}}{\left\{1 + \dfrac{s^2}{k_1 k_2}\left[\dfrac{k_1 y(k_1) - k_2 y(k_2)}{k_1 y(k_2) - k_2 y(k_1)}\right]\right\} - s\,y(s)\left\{\dfrac{1}{k_1 y(k_1)} + \dfrac{1}{k_2 y(k_1)}\left[\dfrac{k_1 y(k_1) - k_2 y(k_2)}{k_1 y(k_2) - k_2 y(k_1)}\right]\right\}}\right]$$

$$= \left[\frac{k_1 y(k_1) - k_2 y(k_2)}{k_1 y(k_2) - k_2 y(k_1)}\right]^2\left[\frac{y(s)\left\{1 + s^2\left[\dfrac{[k_1 y(k_2) - k_2 y(k_1)]}{k_1 k_2[k_1 y(k_1) - k_2 y(k_2)]}\right]\right\} - s\left\{\dfrac{y(k_1)}{k_1 k_2}\dfrac{\left(k_2[\cancel{k_1 y(k_1)} - k_2 y(k_2)] + k_1[k_1 y(k_2) - \cancel{k_2 y(k_1)}]\right)}{(k_1 y(k_1) - k_2 y(k_2))}\right\}}{\left\{1 + s^2\left[\dfrac{[k_1 y(k_1) - k_2 y(k_2)]}{k_1 k_2[k_1 y(k_2) - k_2 y(k_1)]}\right]\right\} - s\,y(s)\left\{\dfrac{1}{k_1 k_2 y(k_1)}\dfrac{\left[k_2[\cancel{k_1 y(k_2)} - k_2 y(k_1)] + k_1[k_1 y(k_1) - \cancel{k_2 y(k_2)}]\right]}{(k_1 y(k_2) - k_2 y(k_1))}\right\}}\right]$$

$$= \left[\frac{k_1 y(k_1) - k_2 y(k_2)}{k_1 y(k_2) - k_2 y(k_1)}\right]^2\left[\frac{y(s)\left\{1 + s^2\left[\dfrac{[k_1 y(k_2) - k_2 y(k_1)]}{k_1 k_2[k_1 y(k_1) - k_2 y(k_2)]}\right]\right\} - s\left\{\dfrac{y(k_1)y(k_2)}{k_1 k_2}\dfrac{[k_1^2 - k_2^2]}{[k_1 y(k_2) - k_2 y(k_1)]}\right\}}{\left\{1 + s^2\left[\dfrac{[k_1 y(k_1) - k_2 y(k_2)]}{k_1 k_2[k_1 y(k_2) - k_2 y(k_1)]}\right]\right\} - s\,y(s)\left\{\dfrac{1}{k_1 k_2}\cdot\dfrac{[k_1^2 - k_2^2]}{[k_1 y(k_2) - k_2 y(k_1)]}\right\}}\right]$$

This is positive if $\mathrm{Re}\,k_1 > 0$ & $\mathrm{Re}\,k_2 > 0$; $(s-k_1)(s-k_2)$ cancels & if k_1 and/or k_2 is (are) a zero of $\mathrm{Ev}\,y(s)$ then $(s+k_1)$ and/or $(s+k_2)$ also cancel.
$\therefore$ if $k_2 = k_1^*$, $\mathrm{Ev}\,y(k_1) = 0$, $\mathrm{Re}\,k_1 > 0$, $\mathrm{Im}\,k_1 \neq 0$, $y_{RR}(s)$ is PR, $\delta[y_{RR}] = \delta[y_R] - 2$

$$\begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}, \quad \begin{bmatrix} v_1 \\ i_1 \end{bmatrix} = \begin{bmatrix} \mathcal{Y}_{11} & \mathcal{Y}_{12} \\ \mathcal{Y}_{21} & \mathcal{Y}_{22} \end{bmatrix} \begin{bmatrix} v_2 \\ -i_2 \end{bmatrix}$$

$\Downarrow$

$$-i_2 = -y_{21} v_1 - y_{22} v_2$$

$$\Rightarrow \quad v_1 = \frac{-1}{y_{21}} \left[-i_2 + y_{22} v_2 \right]$$

$$\Rightarrow \quad i_1 = y_{11} v_1 + y_{12} v_2$$

$$= y_{11} \left\{ -\frac{1}{y_{21}} \left[-i_2 + y_{22} v_2 \right] \right\} + y_{12} v_2$$

$$= \left(y_{12} - \frac{y_{11} y_{22}}{y_{21}} \right) v_2 - \frac{y_{11}}{y_{21}} (-i_2)$$

$$\Rightarrow \begin{bmatrix} v_1 \\ i_1 \end{bmatrix} = \begin{bmatrix} -y_{22}/y_{21} & -1/y_{21} \\ -\Delta y/y_{21} & -y_{11}/y_{21} \end{bmatrix} \begin{bmatrix} v_2 \\ -i_2 \end{bmatrix}$$

$$\mathcal{Y} = -\frac{1}{y_{21}} \begin{bmatrix} y_{22} & 1 \\ \Delta y & y_{11} \end{bmatrix}$$

$\searrow$

$$v_1 = \mathcal{Y}_{11} v_2 - \mathcal{Y}_{12} i_2$$

$$\Rightarrow \quad i_2 = \frac{1}{\mathcal{Y}_{12}} \left[\mathcal{Y}_{11} v_2 - v_1 \right]$$

$$\Rightarrow \quad i_1 = \mathcal{Y}_{21} v_2 - \mathcal{Y}_{22} i_2$$

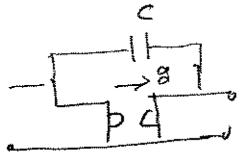
$$= \mathcal{Y}_{21} v_2 - \mathcal{Y}_{22} \left\{ \frac{\mathcal{Y}_{11}}{\mathcal{Y}_{12}} v_2 - \frac{1}{\mathcal{Y}_{12}} v_1 \right\}$$

$$= \frac{\mathcal{Y}_{22}}{\mathcal{Y}_{12}} v_1 - \frac{\Delta \mathcal{Y}}{\mathcal{Y}_{12}} v_2$$

$$\Rightarrow \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} \mathcal{Y}_{22}/\mathcal{Y}_{12} & -\Delta\mathcal{Y}/\mathcal{Y}_{12} \\ -1/\mathcal{Y}_{12} & \mathcal{Y}_{11}/\mathcal{Y}_{12} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$Y = \frac{1}{\mathcal{Y}_{12}} \begin{bmatrix} \mathcal{Y}_{22} & -\Delta \mathcal{Y} \\ -1 & \mathcal{Y}_{11} \end{bmatrix}$$

For



$$Y = \begin{bmatrix} sC & -sC - g \\ -sC + g & sC \end{bmatrix}, \quad \Delta y = g^2 \Rightarrow \mathcal{Y} = -\frac{1}{g - sC} \begin{bmatrix} sC & 1 \\ g^2 & sC \end{bmatrix}$$

for cascade of two such

$$\therefore \quad \mathcal{Y} = \mathcal{Y}_1 \mathcal{Y}_2 = \frac{1}{(g_1 - sC_1)(g_2 - sC_2)} \begin{bmatrix} sC_1 & 1 \\ g_1^2 & sC_1 \end{bmatrix} \begin{bmatrix} sC_2 & 1 \\ g_2^2 & sC_2 \end{bmatrix} = \frac{1}{(sC_1 - g_1)(sC_2 - g_2)} \begin{bmatrix} s^2 C_1 C_2 + g_2^2 & s(C_1 + C_2) \\ s(C_2 g_1^2 + C_1 g_2^2) & s^2 C_1 C_2 + g \end{bmatrix}$$

$$Y = \begin{bmatrix} \dfrac{s^2 C_1 C_2 + g_1^2}{s(C_1 + C_2)} & -\Delta\mathcal{Y}/\mathcal{Y}_{12} \\ -\dfrac{(sC_1 - g_1)(sC_2 - g_2)}{s(C_1 + C_2)} & \dfrac{s^2 C_1 C_2 + g_2^2}{s(C_1 + C_2)} \end{bmatrix}$$

$$; \quad \frac{\Delta\mathcal{Y}}{\mathcal{Y}_{12}} = \frac{(s^2 C_1 C_2 + g_2^2)(s^2 C_1 C_2 + g_1^2) - s^2 (C_1 + C_2)(C_2 g_1^2 - C_1 \ldots}{\left[(sC_1 - g_1)(sC_2 - g_2)\right]^2 \cdot \dfrac{s(C_1 + C_2)}{(sC_1 - g_1)(sC_2 - g_2)}}$$

$$= \frac{\Delta\mathcal{Y}_1 \Delta\mathcal{Y}_2}{\mathcal{Y}_{12}} = \frac{\left[s^2 C_1^2 - g_1^2\right]\left[s^2 C_2^2 - g_2^2\right]}{(g_1 - sC_1)^2 (g_2 - sC_2)^2 \cdot \dfrac{s(C_1 + C_2)}{(g_1 - sC_1)(g_2 - sC_2)}}$$

$$= \frac{(sC_1 + g_1)(sC_2 + g_2)}{s(C_1 + C_2)}$$

$$= \frac{1}{s(C_1 + C_2)} \begin{bmatrix} s^2 C_1 C_2 + g_1^2 & -(sC_1 + g_1)(sC_2 + g_2) \\ -(sC_1 - g_1)(sC_2 - g_2) & s^2 C_1 C_2 + g_2^2 \end{bmatrix} = \frac{1}{s(C_1 + C_2)} \begin{bmatrix} s^2 C_1 C_2 + g_1^2 & -s^2 C_1 C_2 - (g_1 C_2 + g_2 C_1)s - g_1 g_2 \\ -s^2 C_1 C_2 + (g_1 C_2 + g_2 C_1)s - g_1 g_2 & s^2 C_1 C_2 + g_2^2 \end{bmatrix}$$

For Richards' functions $\quad C_1 = y(k_1)/k_1 \quad g_1 = y(k_1)\,; \quad C_2 = y_R(k_2)/k_2\,, \quad g_2 = y_R(k_2)$

$$y_R(k_2) = y(k_1) \left[\frac{k_1 y(k_1) - k_2 y(k_2)}{k_1 y(k_2) - k_1 y(k_1)} \right]$$

$$\therefore \quad C_1 + C_2 = y(k_1) \left\{ \frac{1}{k_1} + \frac{1}{k_2} \left[\frac{k_1 y(k_1) - k_2 y(k_2)}{k_1 y(k_2) - k_2 y(k_1)} \right] \right\}$$

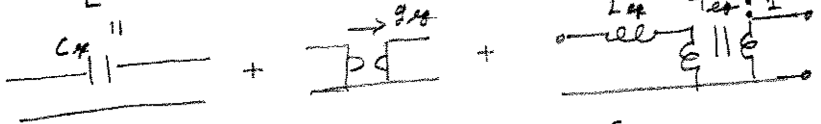
$$= \frac{y^2(k_1)}{k_1 k_2} \left\{ \frac{k_1^2 - k_2^2}{k_1 y(k_2) - k_2 y(k_1)} \right\}$$

$$C_1 C_2 = \frac{y^2(k_1)}{k_1 k_2} \left\{ \frac{k_1 y(k_1) - k_2 y(k_2)}{k_1 y(k_2) - k_2 y(k_1)} \right\}$$

$$g_1 C_2 + g_2 C_1 = \frac{y(k_1)\, y(k_1)}{k_2} \left[\frac{k_1 y(k_1) - k_2 y(k_2)}{k_1 y(k_2) - k_2 y(k_1)} \right] + \frac{y(k_1) y(k_1)}{k_1} \left[\frac{k_1 y(k_1) - k_2 y(k_2)}{k_1 y(k_2) - k_2 y(k_1)} \right] = \frac{y^2(k_1)}{k_1 k_2} (k_1 + k_2) \left[\frac{k_1 y(k_1) - k_2 y(k_2)}{k_1 y(k_2) - k_2 y(k_1)} \right]$$

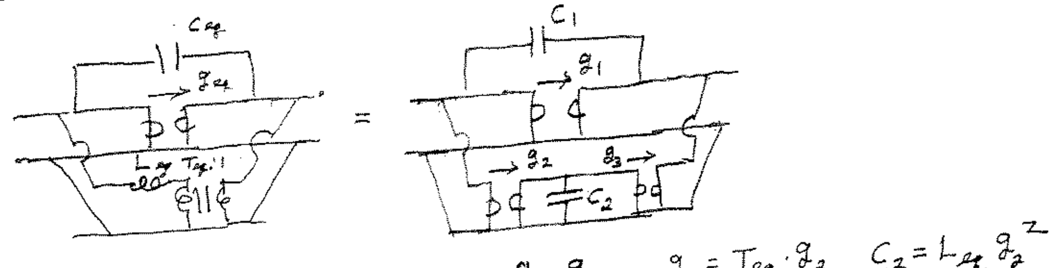
$$\therefore\; Y(s) = s\,\frac{C_1C_2}{C_1+C_2}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} + \frac{(g_1C_2+g_2C_1)}{C_1+C_2}\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix} + \frac{1}{s\left(\frac{C_1+C_2}{g_1^2}\right)}\begin{bmatrix}1 & -(g_2/g_1)\\ -\left(\frac{g_2}{g_1}\right) & (g_2/g_1)^2\end{bmatrix}$$

$$= s\,C_{eq}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} + g_{eq}\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix} + \frac{1}{s\,L_{eq}}\begin{bmatrix}1 & -T_{eq}\\ -T_{eq} & T_{eq}^2\end{bmatrix}$$



(+ = parallel connection)

$\Rightarrow$



$C_1 = C_{eq}$, $g_1 = g_{eq}$, $g_3 = T_{eq}\cdot g_2$, $C_2 = L_{eq}\, g_2^2$
with g_2 free to choose.

$$C_{eq} = \frac{C_1C_2}{C_1+C_2} = \frac{\frac{y^2(k_1)}{k_1k_2}\cdot\left[\frac{k_1y(k_1)-k_2y(k_2)}{k_1y(k_2)-k_2y(k_1)}\right]}{\frac{y^2(k_1)}{k_1k_2}\left[\frac{k_1^2-k_2^2}{k_1y(k_2)-k_2y(k_1)}\right]} = \frac{k_1y(k_1)-k_2y(k_2)}{k_1^2-k_2^2}$$

$$L_{eq} = \frac{C_1+C_2}{g_1^2} = \frac{1}{k_1k_2}\left[\frac{k_1^2-k_2^2}{k_1y(k_2)-k_2y(k_1)}\right]$$

$$T_{eq} = \frac{g_2}{g_1} = \left[\frac{k_1y(k_1)-k_2y(k_2)}{k_1y(k_2)-k_2y(k_1)}\right]$$

$$g_{eq} = \frac{g_1C_2+g_2C_1}{C_1+C_2} = \frac{k_1y(k_1)-k_2y(k_2)}{k_1-k_2}$$

For $k_2 = k_1^*$ and $y(s)$ PR, $y(k_2) = y(k_1^*) = y^*(k_1)$

$$C_{eq} = \frac{k_1y(k_1)-(k_1y(k_1))^*}{k_1^2-(k_1^2)^*} = \frac{2j\,\mathrm{Im}(k_1y(k_1))}{2j\,\mathrm{Im}(k_1^2)} = \frac{\mathrm{Im}(k_1y(k_1))}{\mathrm{Im}(k_1^2)}$$

$$L_{eq} = \frac{1}{|k_1|^2}\,\frac{\mathrm{Im}(k_1^2)}{\mathrm{Im}(k_1y^*(k_1))}$$

$$T_{eq} = \mathrm{Im}(k_1y(k_1))/\mathrm{Im}(k_1y^*(k_1))$$

$$g_{eq} = \mathrm{Im}(k_1y(k_1))/\mathrm{Im}(k_1)$$

These are real & $C_{eq}, L_{eq} > 0$ if $y(s)$ is PR

Using $y_L = \frac{y_{22}\,y - \Delta y}{y_{11} - y}$ and dividing numerator and denominator of y_{RR} by the denominator coefficient of $-y$, that is by $s\left\{\frac{1}{k_1k_2}\,\frac{k_1^2-k_2^2}{k_1y(k_2)-k_2y(k_1)}\right\}$ shows that the $Y(s)$ matrix above agrees with that in y_{RR}, as it should, and that $y_L(s) = y_{RR}(s)/\left[\frac{k_1y(k_1)-k_2y(k_2)}{k_1y(k_2)-k_2y(k_1)}\right]^2$, In the case of $k_2 = k_1^*$, $y_L(s) = y_{RR}(s)/\left[\frac{\mathrm{Im}(k_1y(k_1))}{\mathrm{Im}(k_1y^*(k_1))}\right]^2$. The 2-port above is lossless, so repetition shows that a PR y can be synthesized by a lossless 2-port terminated in one resistor (we can handle complex k by L'Hospital's rule & real k as before)