Semistate equations; linear time-invariant

EE 610
10/15/07

$$E\frac{dx}{dt} = Ax + Bu$$
$$y = Cx$$

x = semi-state
u = input vector
y = output
A, B, C, E = constant matrices

$$E\,\mathcal{L}\left[\frac{dx}{dt}\right] = sE\,X(s) = A\,\mathcal{L}[x] + B\,\mathcal{L}[u] = AX(s) + BU(s)$$

$$\mathcal{L}[y] = C\,\mathcal{L}[x]$$
$$Y(s) = CX(s)$$

$$(sE - A)X(s) = BU(s)$$
or $X(s) = [sE - A]^{-1}BU(s)$

$$Y(s) = C[sE - A]^{-1}B \cdot U(s) \Rightarrow T(s) = C[sE - A]^{-1}B$$
= transfer function matrix

Ex:

$$\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\frac{dx}{dt} = \begin{bmatrix}0 & -3\\ 4 & 0\end{bmatrix}x + \begin{bmatrix}1 & -2\\ 0 & 5\end{bmatrix}\begin{bmatrix}u_1\\ u_2\end{bmatrix}$$

$$y = \begin{bmatrix}y_1\\ y_2\end{bmatrix} = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}x$$

$$T(s) = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\left[\begin{bmatrix}s\cdot 1 & 0\\ 0 & 0\end{bmatrix} - \begin{bmatrix}0 & -3\\ 4 & 0\end{bmatrix}\right]^{-1}\begin{bmatrix}1 & -2\\ 0 & 5\end{bmatrix} = 1_2\begin{bmatrix}s & 3\\ -4 & 0\end{bmatrix}^{-1}\begin{bmatrix}1 & -2\\ 0 & 5\end{bmatrix}$$

$$= \frac{1}{12}\cdot 1_2\begin{bmatrix}0 & -3\\ 4 & s\end{bmatrix}\begin{bmatrix}1 & -2\\ 0 & 5\end{bmatrix} = \frac{1}{12}\begin{bmatrix}0 & -15\\ 4 & 5s-8\end{bmatrix}$$

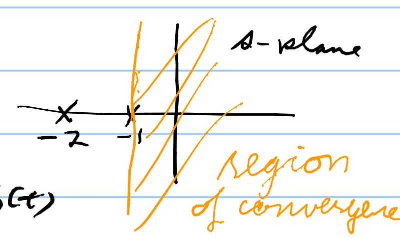
has a pole at ∞

if we had $\dot{x}_s = A_s x_s + B_s u$, $y = C_s x_s + D_s u$ $\Rightarrow T_s(s) = D_s + C_s(s1 - A_s)^{-1}B_s$

$$T_s(\infty) = D_s$$

as we had a pole @ ∞ can not get these state equations from the semi-state ones.

$$T(s) = \frac{2s}{(s+1)(s+2)} = \frac{-2}{(s+1)} + \frac{4}{(s+2)} + 5s$$



$Y(s)$ $T(s)\times 1$ = impulse response transform; $u(t) = \delta(t)$

$y_\delta(t) = -2e^{-t}1(t) + 4e^{-2t}1(t)$ $+ 5\frac{d\delta(t)}{dt}$ for the given region of convergence

To get time-domain response for state variable

$$T(s) = D + C(s1_k - A)^{-1}B$$ k = size of state x

$$(s1_k - A)^{-1} = \left(s\left[1_k - \frac{A}{s}\right]\right)^{-1} = \frac{1}{s}\cdot\frac{1}{1_k - \frac{A}{s}} \approx \frac{1}{s}\left[1_k + \frac{A}{s} + \frac{A^2}{s^2} + \dots\right]$$

$$\frac{dx}{dt} = Ax + Bu \Rightarrow \frac{dx}{dt} = Ax \text{ ; try } x(t) = e^{At}x_0$$

here $x_0 = x(0)$

$$e^{at} = 1 + at + \frac{(at)^2}{2!} + \dots$$

$$\Downarrow$$

$$e^{At} = 1_k + At + \frac{(At)^2}{2!} + \dots = \sum_{i=0}^{\infty} \frac{A^i t^i}{i!}$$

$$s1_k\mathcal{L}[x] - x_0 = A\mathcal{L}[x]$$

$$\mathcal{L}[x] = (s1_k - A)^{-1}x_0$$

$$\mathcal{L}((s1_k - A)^{-1}) = \mathcal{L}[e^{At}]$$

$$\frac{de^{At}}{dt} = \frac{d\sum_{i=0}^{\infty} A^i t^i}{dt} = \sum_{i=0}^{\infty} \frac{A^i i t^{i-1}}{i!} = A\sum_{i=1}^{\infty} \frac{A^{i-1}t^{i-1}}{i!}\cdot i$$

$$= A\sum_{j=0}^{\infty} \frac{A^j t^j}{j!} = Ae^{At} \qquad j = i-1$$

If x is a k-vector, $k < \infty$

then $(s1_k - A)^{-1}$ is a rational matrix

so can make a partial fraction expansion of each term

to avoid the infinite series.

Ex: $A = \begin{bmatrix} 1 & -1 \\ 3 & 4 \end{bmatrix}$, $e^{At} = \mathcal{L}[(s1_2 - A)^{-1}]$

$$s1_2 - A = \begin{bmatrix} s-1 & \cancel{s}+1 \\ \cancel{3}-3 & s-4 \end{bmatrix} = s\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{=1_2} - \begin{bmatrix} 1 & -1 \\ 3 & 4 \end{bmatrix} \text{ ; } \det = (s-1)(s-4)+3$$

$$= s^2 - 5s + 4 + 3$$

$$\mathrm{Den}(s) = s^2 - 5s + 7 \text{ ; } \quad s_{1,2} = \frac{5}{2} \pm \sqrt{(5/2)^2 - 7} = \frac{5}{2} \pm \frac{1}{2}\sqrt{-3}$$

$$= \left(s - \tfrac{1}{2}(5 + j\sqrt{3})\right)\left(s - \tfrac{1}{2}(5 - j\sqrt{3})\right)$$

$$N_{mn}(s) = \begin{bmatrix} s-4 & -1 \\ 3 & s-1 \end{bmatrix}$$

$$(sI_2 - A)^{-1} = \frac{1}{s^2 - 5s + 7} \begin{bmatrix} s-4 & -1 \\ 3 & s-1 \end{bmatrix}$$

$$= \frac{1}{\left(s - \frac{1}{2}(5 + j\sqrt{3})\right)} \begin{bmatrix} & \\ & \end{bmatrix} + \frac{1}{\left(s - \frac{1}{2}(5 - j\sqrt{3})\right)} \begin{bmatrix} & \\ & \end{bmatrix}$$