solutions 610 final 12/18/07

#1.

$$Y_{indef\,gy} = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ 1 & 0 & -g & g & \\ 2 & g & 0 & -g & \\ 3 & -g & g & 0 & \\ 4 & & & & \end{array} \qquad Y_{indef\,LRC} = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ & \frac{1}{sL} & -\frac{1}{sL} & & \\ & -\frac{1}{sL} & \frac{1}{sL}+sC & & -sC \\ & & & G & -G \\ & & -sC & -G & G+sC \end{array}$$

a1) $$Y_{ind} = \begin{bmatrix} \frac{1}{sL} & -g-\frac{1}{sL} & g & 0 \\ g-\frac{1}{sL} & \frac{1}{sL}+sC & -g & -sC \\ -g & g & G & -G \\ 0 & -sC & -G & G+sC \end{bmatrix}$$

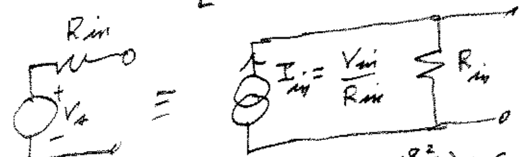
a2) remove row & column 4

$$Y_{def} = \begin{bmatrix} \frac{1}{sL} & -g-\frac{1}{sL} & g \\ g-\frac{1}{sL} & \frac{1}{sL}+sC & -g \\ -g & g & G \end{bmatrix} \Rightarrow \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = Y_{def} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}$$

set $I_3 = 0 \Rightarrow GV_3 = -\begin{bmatrix} -g & g \end{bmatrix}\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \Rightarrow V_3 = \begin{bmatrix} \frac{g}{G} & -\frac{g}{G} \end{bmatrix}\begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{sL} & -g-\frac{1}{sC} \\ g-\frac{1}{sL} & \frac{1}{sL}+sC \end{bmatrix}\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} + \begin{bmatrix} g \\ -g \end{bmatrix}\begin{bmatrix} g/G & -g/G \end{bmatrix}\begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

$$\Rightarrow Y(s) = \begin{bmatrix} \frac{1}{sL}+\frac{g^2}{G} & -g-\frac{1}{sL}-g^2/G \\ g-\frac{1}{sL}-\frac{g^2}{G} & \frac{1}{sL}+sC+\frac{g^2}{G} \end{bmatrix}$$

b1)  $\Rightarrow$ add G_{in} to $Y_{11}(s)$, $G_{in} = 1/R_{in}$

$$\begin{bmatrix} I_{in} \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{sL}+(\frac{g^2}{G})+G_{in} & -g-\frac{1}{sL}-g^2/G \\ g-\frac{1}{sL}-g^2/G & \frac{1}{sL}+sC+g^2/G \end{bmatrix}\begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

$I_{out\,load} \Rightarrow$

$\Rightarrow V_1 = -(\frac{1}{sL}+sC+g^2/G)V_2/(g-\frac{1}{sL}-g^2/G)$

$$\Rightarrow \frac{I_{in}}{V_2} = \frac{V_{in}/R_{in}}{V_2} = (\frac{1}{sL}+(\frac{g^2}{G})+G_{in})\left[-\frac{(\frac{1}{sL}+sC+\frac{g^2}{G})}{g-\frac{1}{sL}-g^2/G}\right] + \left[-g-\frac{1}{sL}-g^2/G\right]$$

$$\Rightarrow \frac{1}{T(s)} = R_{in}\left\{\frac{-(\frac{1}{sL}+g^2/G+G_{in})(\frac{1}{sL}+sC+g^2/G)+[g-\frac{1}{sL}-g^2/G][-g-\frac{1}{sL}-\frac{g^2}{G}]}{g-\frac{1}{sL}-g^2/G}\right\}$$

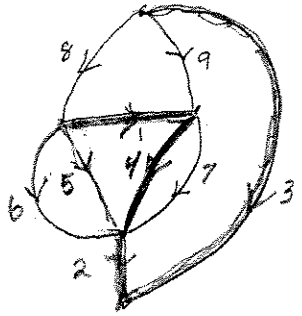
$$\Rightarrow T(s) = G_{in}\{sL(g-g^2/G)-1\}/\{sL[-(\tfrac{1}{sL}+\tfrac{g^2}{G})^2-(G_{in}+sC)(\tfrac{1}{sL}+\tfrac{g^2}{G})-G_{in}sC-g^2+(\tfrac{1}{sL}+\tfrac{g^2}{G})^2]$$

$$= -G_{in}\{sL(g-g^2/G)-1\}/\{CL(g^2/G+G_{in})s^2+(C+LG_{in}\tfrac{g^2}{G}+Lg^2)s+G_{in}\}$$

b2) If all of G_{in}, C, L, G, $g > 0$ all poles are in Re $s<0$. The numerator has all coefficients >0 if & only if $1 < g/G$ and then $T(s) = \frac{as+b}{cs^2+ds+e}$ with all coefficients >0 & hence PR. But $T(s)$ is not PR if $G > g > 0$ as then currents can flow down through C causing a negative $T(\sigma)$ for some σ in Re $s>0$. If $g<0$ then the numerator - signs as well and $T(s)$ is PR

#2

a)



darker lines = tree, $t=4$, $b=9$, $\ell=b-t=5$

cutset: $0=\mathcal{C}i_b \Rightarrow \mathcal{C}$ is 4×9

$$\mathcal{C}=\begin{array}{c} \begin{array}{ccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{array} & \\ \left[\begin{array}{ccccccccc} 1 & 0 & 0 & 0 & 1 & 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & -1 & -1 \end{array}\right] & \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} \end{array}$$

Tie set: $0=\mathcal{T}v_b \Rightarrow \mathcal{T}$ is 5×9

$$\mathcal{T}=\begin{array}{c} \begin{array}{ccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{array} & \\ \left[\begin{array}{ccccccccc} -1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & -1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right] & \begin{array}{c} 1 \Leftarrow 5 \\ 2 \Leftarrow 6 \\ 3 \Leftarrow 7 \\ 4 \Leftarrow 8 \\ 5 \Leftarrow 9 \end{array} \end{array}$$

; $x=\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ i_5 \\ i_6 \\ i_7 \\ i_8 \\ i_9 \end{bmatrix}$

b) Components

$i_{1b}=sC_1v_{1b} \Rightarrow$ capacitor $\quad -x_5-x_6+x_8=sC_1x_1$

$i_{2b}=I_{b2}$, $v_{3b}=V_{dd3}$ } independent sources $\Rightarrow x_8+x_9=I_{b2}$; $x_2=V_{dd}$

$i_{4b}=0$, $i_{b5}=0$ } gate currents $\quad -x_5-x_6-x_7+x_8+x_9=0$; $x_5=0$

$i_{6b}=f(v_{4b})$, $i_{7b}=f(v_{5b})$ } drain currents $\quad x_6=f(x_4)$, $x_7=f(x_1+x_4)$

$v_{8b}=sL_8i_{8b}$, $v_{9b}=sL_9i_{9b}$ } inductor $\quad -x_1-x_2+x_3-x_4=sL_8x_8$

$-x_2+x_3-x_4=sL_9x_9$

Into matrix form these are

$$\begin{array}{cc} & \begin{array}{ccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \end{array} & \left[\begin{array}{ccccccccc} C_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & L_8 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & L_9 \end{array}\right] \end{array} \frac{dx}{dt} = \begin{array}{c} \begin{array}{ccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{array} \\ \left[\begin{array}{ccccccccc} 0 & 0 & 0 & 0 & -1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & -1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \end{array} x + \begin{bmatrix} 0 \\ -I_{b2} \\ -V_{dd3} \\ 0 \\ 0 \\ -f(x_4) \\ -f(x_1+x_4) \\ 0 \\ 0 \end{bmatrix}$$

c) The equations of McGovern's handout linearize $f(v_{ga})=g_m v_{ga}$. Doing that we have $E\,dx/dt=Ax$ with $x(0)$ as initial conditions, these latter being $v_{1b}(0)$, $i_{8b}(0)$, $i_{9b}(0)$. Then we need to add the excitation term. Once that is done we reduce the above equations to one ODE. Note that the system appears to be degree 3 while paper only has degree 2.

#3. a) $Y_{2\text{-port}} = \begin{bmatrix} G & -g-G \\ g-G & G \end{bmatrix}$; $\begin{bmatrix} i_1 \\ i_2 = -y_L v_2 \end{bmatrix} = \begin{bmatrix} G & -g-G \\ g-G & G \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$

$$\Rightarrow v_2 = [-(g-G)v_1/(G+y_L)]v_1$$

$$\Rightarrow \frac{i_1}{v_1} = G - \frac{(-g-G)(g-G)}{G+y_L} = \frac{G^2+Gy_L-[-g^2+G^2]}{G+y_L} = \frac{g^2+Gy_L}{G+y_L} = y_{in}$$

$$\Rightarrow g^2+Gy_L = Gy_{in} + y_{in}y_L \Rightarrow g^2 - Gy_{in} = (y_{in}-G)y_L$$

$$\Rightarrow y_L(s) = [g^2 - Gy_{in}(s)]/[y_{in}-G]$$

b) Let $y_{in}(s) = (\sum_{i=0}^{m} a_i s^i)/(\sum_{i=0}^{m} b_i s^i)$ with $a_m, b_m, a_0, b_0 \neq 0$, $m>1$

$$\therefore y_L(s) = \left(\sum_{i=0}^{m}(g^2 b_i - G a_i)s^i\right)\Big/\left(\sum_{i=0}^{m}(a_i - G b_i)s^i\right)$$

We desire a zero at ∞

$$\Rightarrow g^2 b_m - G a_m = 0 \Rightarrow G = g^2(b_m/a_m) \quad (\& \ a_m \neq G b_m \Rightarrow G \neq a_m/b_m)$$

and also a pole at 0

$$\Rightarrow a_0 - G b_0 = 0 \Rightarrow G = (a_0/b_0) \Rightarrow g^2 = \left(\frac{a_0 a_m}{b_0 b_m}\right)$$

$$(\& \ g^2 b_0 \neq G a_0) \Rightarrow g^2 \neq (a_0^2/b_0^2)$$

Since all coefficients are different, only the numerator mth and the denominator 0th coefficients will be zero. This gives a simple pole at $s=0$ (and a simple zero at $s=\infty$), if also $G \neq a_m/b_m$ & $g^2 \neq (a_0/b_0)^2$

For $y_L(s)$ we need the residue at $s=0$ to be positive. This residue, r_0, is

$$r_0 = (g^2 b_0 - G a_0)/(a_1 - G b_1)$$

$$= \left(\frac{a_0 a_m}{b_0 b_m}\cdot b_0 - \left(\frac{a_0}{b_0}\right)a_0\right)\Big/\left(a_1 - \left(\frac{a_0}{b_0}\right)b_1\right)$$

But r_0 can be negative if $\frac{a_m}{b_m} < \frac{a_0}{b_0}$ & $\frac{a_1}{b_1} > \frac{a_0}{b_0}$. This is possible in the PR y_{in}

$$y_{in} = \frac{\frac{7}{8}s}{2s+3} + \frac{1}{s+2} = \frac{\frac{7}{8}s^2 + (2+7/4)s + 3}{2s^2+(3+4)s+6}$$

since $\frac{a_m}{b_m} = \frac{7}{16} < \frac{3}{6} = \frac{1}{2} = \frac{a_0}{b_0} < \frac{a_1}{b_1} = \frac{15/4}{7} = \frac{15}{28} = \frac{14+1}{28} = \frac{1}{2} + \frac{1}{28}$

#4. a) $y(s) = \dfrac{s(4s + 27/4)}{(s+1)(s+2)} = \dfrac{4s(s + 27/16)}{(s+1)(s+2)}$

poles: $s_{p_1} = -1$, $s_{p_2} = -2$ } poles & zeros alternate

zeros: $s_{z_1} = 0$, $s_{o_2} = -27/16 = -1\tfrac{11}{16}$

b) Let $y(s) = k_\infty s + k_0 + \sum_{i=1}^{m} \dfrac{k_i s}{s+\sigma_i}$

$$2\,\mathrm{Ev}\,y(s) = \left[k_\infty s + k_0 + \sum_{i=1}^{m} \frac{k_i s}{s+\sigma_i}\right] + \left[-k_\infty s + k_0 + \sum_{i=1}^{m} \frac{-k_i s}{-s+\sigma_i}\right]$$

$$= 2k_0 + \sum_{i=1}^{m} \left(\frac{k_i s(-s+\sigma_i) + (-k_i s)(s+\sigma_i)}{(s+\sigma_i)(-s+\sigma_i)}\right)$$

$$= 2k_0 + \sum_{i=1}^{m} \frac{-2k_i s^2}{(s+\sigma_i)(-s+\sigma_i)}$$

$\therefore$ $\mathrm{Ev}\,y(s) = 0$ $\Rightarrow$ $k_0 = \sum_{i=1}^{m} \dfrac{k_i s^2}{(s+\sigma_i)(-s+\sigma_i)} = f(s^2)$

The derivative on the right is

$$\frac{d f(\sigma^2)}{d\sigma} = \sum_{i=1}^{m} \left[\frac{2k_i s}{(-s^2+\sigma_i^2)} - \frac{k_i s^2(-2s)}{(-s^2+\sigma_i^2)^2}\right]\Bigg|_{s=\sigma}$$

$$= \sum_{i=1}^{m} \frac{2k_i s(-s^2+\sigma_i^2) + 2k_i s^3}{(-s^2+\sigma_i^2)^2}\Bigg|_{s=\sigma}$$

$$= \sum_{i=1}^{m} \frac{2k_i \sigma_i^2 \sigma}{(-\sigma^2+\sigma_i^2)^2} \geq 0$$ $\Rightarrow$ $f(\sigma^2)$ increasing with poles @ σ_i in $\sigma > 0$.

$\therefore$ $f(\sigma^2)$ passes through $k_0 \geq 0$ m times in $\sigma > 0$.

If $y(s) = k_\infty s$, every s is a zero of $\mathrm{Ev}\,y$

If $y(s) = k_\infty s + k_0$, $k_0 > 0$ no s is a zero of $\mathrm{Ev}\,y$

If $y(s) = k_\infty s + k_0 + \sum_{i=1}^{m} \dfrac{k_i s}{s+\sigma_i}$, $k_i > 0$ there are m zeros of $\mathrm{Ev}\,y(s)$ in $\sigma > 0$ & m in $\sigma < 0$.

(if $k_0 = 0$ there are 2 @ $\sigma = 0$, $m-1$ in $\sigma > 0$ & in $\sigma < 0$