# Question 2

## Part (A)

$$AB = \begin{bmatrix} -8 & -2a & 4b+2a \\ 0 & -4 & 2 \\ 0 & 0 & -4 \end{bmatrix} \qquad BA = \begin{bmatrix} -8 & -2a & 2b \\ 0 & -4 & 2 \\ 0 & 0 & -4 \end{bmatrix}$$

if $AB = BA$

$\therefore \; a = -b$

## Part (B)

$a = 1 \Rightarrow b = -1$

$$A = \begin{bmatrix} 4 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\det(\lambda \overset{\text{identity matrix}}{I} - A) = 0 \Rightarrow \begin{bmatrix} \lambda-4 & -1 & 0 \\ 0 & \lambda-2 & -1 \\ 0 & 0 & \lambda-2 \end{bmatrix} = 0$$

$$(\lambda-4)(\lambda-2)(\lambda-2) = 0$$

$$\boxed{\begin{aligned} \lambda_1 &= 4 \\ \lambda_2 &= 2 \\ \lambda_3 &= 2 \end{aligned}}$$

$$B = \begin{bmatrix} -2 & 0 & -1 \\ 0 & -2 & 2 \\ 0 & 0 & -2 \end{bmatrix}$$

$$\det(\lambda I - B) = 0 \Rightarrow \begin{bmatrix} \lambda+2 & 0 & 1 \\ 0 & \lambda+2 & -2 \\ 0 & 0 & \lambda+2 \end{bmatrix} = 0$$

$$(\lambda+2)(\lambda+2)(\lambda+2) = 0$$

$\lambda_1 = -2$
$\lambda_2 = -2$
$\lambda_3 = -2$

c)

$$e^A = \alpha_0 + \alpha_1 A + \alpha_2 A^2$$

$$e^{\lambda_1} = \alpha_0 + \alpha_1 \lambda_1 + \alpha_2 (\lambda_1)^2$$

$$e^{\lambda_2} = \alpha_0 + \alpha_1 \lambda_2 + \alpha_2 (\lambda_2)^2$$

$$\frac{d}{d\lambda_2}(e^{\lambda_2}) = \frac{d}{d\lambda_2}(\alpha_0 + \alpha_1\lambda_2 + \alpha_2\lambda_2^2) \Longrightarrow e^{\lambda_2} = \alpha_1 + 2\alpha_2\lambda_2$$

$$\Longrightarrow \begin{cases} \alpha_0 + \alpha_1\lambda_1 + \alpha_2\lambda_1^2 = e^{\lambda_1} \\ \alpha_0 + \alpha_1\lambda_2 + \alpha_2\lambda_2^2 = e^{\lambda_2} \\ \alpha_1 + 2\alpha_2\lambda_2 = e^{\lambda_2} \end{cases} \Longrightarrow \alpha_0 = \frac{\begin{vmatrix} e^{\lambda_1} & \lambda_1 & \lambda_1^2 \\ e^{\lambda_2} & \lambda_2 & \lambda_2^2 \\ e^{\lambda_2} & 1 & 2\lambda_2 \end{vmatrix}}{\begin{vmatrix} 1 & \lambda_1 & \lambda_1^2 \\ 1 & \lambda_2 & \lambda_2^2 \\ 0 & 1 & 2\lambda_2 \end{vmatrix}} = \frac{4e^4 - 16e^2}{4} = e^2(e^2 - 4)$$

$$\alpha_1 = \frac{\begin{vmatrix} 1 & e^{\lambda_1} & 16 \\ 1 & e^{\lambda_2} & 4 \\ 0 & e^{\lambda_2} & 4 \end{vmatrix}}{4} = \frac{16e^2 - 4e^4}{4} = e^2(4 - e^2)$$

$$\alpha_2 = \frac{\begin{vmatrix} 1 & 4 & e^4 \\ 1 & 2 & e^2 \\ 0 & 1 & e^2 \end{vmatrix}}{4} = \frac{-3e^2 + e^4}{4} = \frac{e^2}{4}(e^2 - 3)$$

For $B$:

$$e^B = \alpha_0 I + \alpha_1 B + \alpha_2 B^2$$

$$\begin{cases} e^{\lambda} = \alpha_0 + \alpha_1\lambda + \alpha_2\lambda^2 \\ e^{\lambda} = \alpha_1 + 2\alpha_2\lambda \\ e^{\lambda} = 2\alpha_2 \end{cases} \Longrightarrow \alpha_2 = \frac{e^{-2}}{2} \Longrightarrow \alpha_1 = e^{\lambda} - 2\alpha_2\lambda = e^{-2} - e^{-2}(-2) = 3e^{-2}$$

$\alpha_0 = e^{\lambda} - \alpha_1 \lambda - \alpha_2 \lambda^2 = e^{-2} + 2(3e^{-2}) - 2e^{-2} = 5e^{-2}$

$\Longrightarrow e^A = e^2(e^2-4)I_3 + e^2(4-e^2)A + \frac{e^2}{4}(e^2-3)A^2$

$e^B = 5e^{-2}I_3 + 3e^{-2}B + \frac{e^{-2}}{2}B^2$

$e^A \cdot e^B = 5(e^2-4)I_3 + \frac{1}{2}(e^2-4)B^2 + 3(e^2-4)B + 5/4(e^2-3)A^2 + \frac{1}{8}(e^2-3)A^2B^2 + 3/4(e^2-3)A^2B +$
$5(4-e^2)A + \frac{1}{2}(4-e^2)A \cdot B^2 + 3(4-e^2)A \cdot B$

d) using MATLAB $e^A \cdot e^B = \begin{bmatrix} e^2 & \frac{1}{2}e^2 - \frac{1}{2} & \frac{1}{4}e^2 - \frac{7}{4} \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$

using MATLAB $e^{A+B} = \begin{bmatrix} e^2 & \frac{+1}{2}e^2 - \frac{1}{2} & \frac{1}{4}e^2 - \frac{7}{4} \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$

So $e^{A+B} = e^A \cdot e^B$

e) find $C$ such that $e^A \cdot e^B = e^C$

we know that the eigenvalues of $e^C$ are $e^2, 1, 1$ then $e^C = \alpha_0 I_3 + \alpha_1 C + \alpha_2 C^2$

$$\begin{cases} e^2 = \alpha_0 + \alpha_1 e^2 + \alpha_4 e^4 \\ e = \alpha_0 + \alpha_1 + \alpha_2 \\ e = \alpha_1 + 2\alpha_2 \end{cases} \Longrightarrow e^C - \alpha_0 I_3 = \alpha_1 C + \alpha_2 C^2$$

if $C = \begin{bmatrix} l_1 & l_2 & l_4 \\ 0 & l_3 & l_5 \\ 0 & 0 & l_6 \end{bmatrix} \Rightarrow e^A \cdot e^B - \alpha_0 I_3 = \alpha_1 C + \alpha_2 C^2$

$\alpha_0, \alpha_1, \alpha_2$ are calculated using MATLAB and the result is as follows:

$$C = \begin{bmatrix} 0 & -1.5 & -1 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix}$$