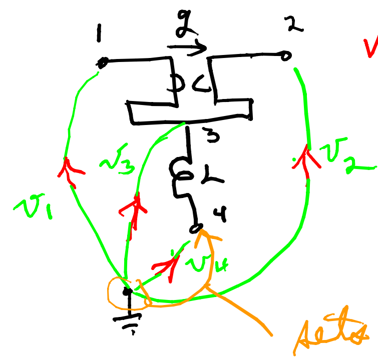
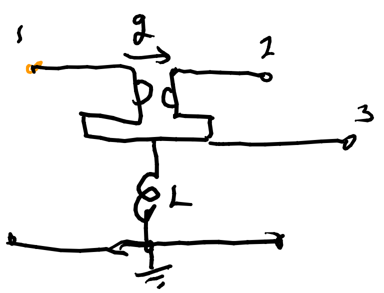


$$V = \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{bmatrix}, \quad \begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \end{bmatrix} = Y_{ind} V$$

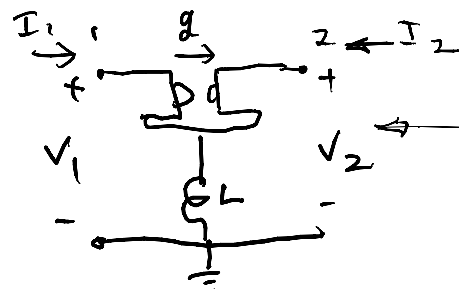
sets $V_4 = 0$ multiplies last column by 0

$$Y_{ind} = \begin{bmatrix} 0 & g & -g & 0 \\ -g & 0 & +g & 0 \\ g & -g & \frac{1}{sL} & -\frac{1}{sL} \\ 0 & 0 & -\frac{1}{sL} & \frac{1}{sL} \end{bmatrix}$$

$$Y_{def} = \begin{bmatrix} 0 & g & -g \\ -g & 0 & +g \\ g & -g & 1/sL \end{bmatrix}; \quad \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = Y_{def} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}$$



if desire the 2-port Y



$I_3 = 0$

$$\left[\begin{array}{c} I_1 \\ I_2 \\ \hline I_3 \end{array}\right] = \left[\begin{array}{c} I_1 \\ I_2 \\ \hline 0 \end{array}\right] = \left[\begin{array}{cc|c} 0 & g & -g \\ -g & 0 & g \\ \hline g & -g & 1/sL \end{array}\right] \left[\begin{array}{c} V_1 \\ V_2 \\ \hline V_3 \end{array}\right]$$

$$\begin{bmatrix} I \\ 0 \end{bmatrix} = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} \begin{bmatrix} V \\ V_3 \end{bmatrix}$$

solve for V_3

$$0 = Y_{21} V + Y_{22} V_3$$

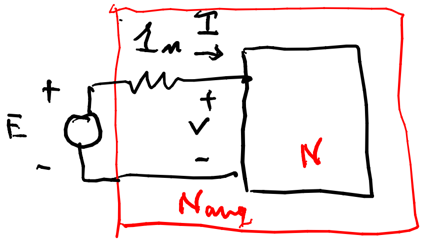
$$\Rightarrow V_3 = -Y_{22}^{-1} Y_{21} V$$

$$I = Y_{11} V + Y_{12} V_3 = [Y_{11} - Y_{12} Y_{22}^{-1} Y_{21}] V$$

$$\therefore \quad I = YV \Rightarrow Y = Y_{11} - Y_{12} Y_{22}^{-1} Y_{21}$$

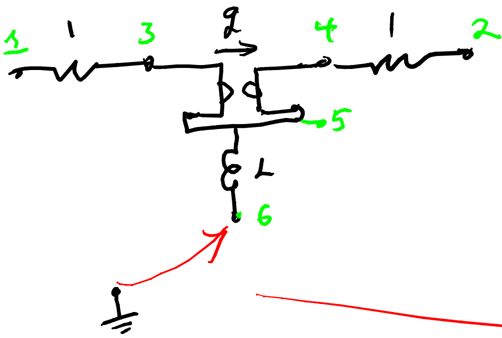
$$= \begin{bmatrix} 0 & g \\ -g & 0 \end{bmatrix} - \begin{bmatrix} -g \\ g \end{bmatrix} [sL] [g \ -g]$$

$$Y = \begin{bmatrix} 0 & g \\ -g & 0 \end{bmatrix} - sL \begin{bmatrix} -g^2 & +g^2 \\ g^2 & -g^2 \end{bmatrix} = \begin{bmatrix} sLg^2 & g - sLg^2 \\ -g - sLg^2 & sLg^2 \end{bmatrix}$$



$$S = 1_n - 2Y_{aug}$$

Ex:



$$Y_{mod} = \begin{bmatrix} 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ -1 & 0 & 1+0 & g & -g & 0 \\ 0 & -1 & -g & 1+0 & +g & 0 \\ 0 & 0 & g & -g & 1/sL & -1/sL \\ 0 & 0 & 0 & 0 & -1/sL & 1/sL \end{bmatrix}$$

gnd node 6 → removes row 6 & col. 6
to get Y_{aug} eliminate nodes 3, 4, 5

$$Y_{aug} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & g & -g \\ -g & 1 & +g \\ g & -g & 1/sL \end{bmatrix}^{-1} \begin{bmatrix} -1 & 0 \\ 0 & -1 \\ 0 & 0 \end{bmatrix}$$

$$\det Y_{22} = \frac{1}{sL} - g^3 + g^3 + g^2 + g^2 + g^2/sL = \frac{1}{sL}(1+g^2) + 2g^2$$

$$\Delta_{11} = \frac{1}{sL} + g^2 \qquad \Delta_{21} = (-1)^{2+1}\left[\frac{g}{sL} - g^2\right]$$

$$\Delta_{12} = (-1)^{1+2}\left[-\frac{g}{sL} - g^2\right] \qquad \Delta_{22} = \frac{1}{sL} + g^2$$

$$\therefore \quad Y_{aug} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \frac{1}{\frac{1}{sL}(1+g^2) + 2g^2} \begin{bmatrix} \frac{1}{sL} + g^2 & -\frac{g}{sL} + g^2 \\ \frac{g}{sL} + g^2 & \frac{1}{sL} + g^2 \end{bmatrix} \quad ; \quad S = 1_2 - 2Y_{aug}$$

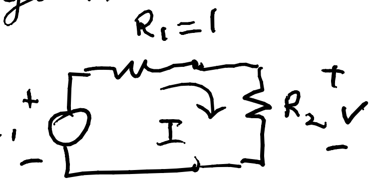
$$S = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} + \frac{2\cancel{sL}}{2g^2Ls + (1+g^2)} \begin{bmatrix} 1 + sg^2L & -g + sg^2L \\ g + sg^2L & 1 + sg^2L \end{bmatrix}$$

this is bounded real as a passive 2-port

should have $S^T(-s) \cdot S(s) = 1_2$ as lossless

Power gain:

$2V_1^i = E_1$



for max power into R_2
desire $R_2 = R_1$ $\Rightarrow$ means reflection coefficient for $R_2 = 0$

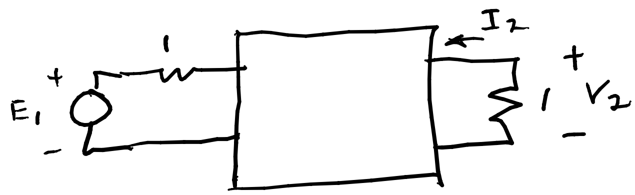
$$I = \frac{E}{R_1 + R_2}, \quad V = \frac{R_2}{R_1+R_2}E, \quad P_{ower} = VI = \frac{R_2}{(R_1+R_2)^2}E^2$$

$\frac{1}{E^2}\frac{dP}{dR_2} = 0$ to maximize:

$$= \frac{1}{(R_1+R_2)^2} - \frac{2\cdot R_2}{(R_1+R_2)^3} = \frac{1}{(R_1+R_2)^3}[R_1+R_2-2R_2] = 0$$

$$\downarrow$$

$$R_2 = R_1$$



$$; P(j\omega) = \mathrm{Re}(\cancel{-}V_2^*(j\omega)\, I_2(j\omega))$$
$$= \mathrm{Re}(V_2^{r*}(j\omega)\, V_2^r(j\omega))$$
$$= |V_2^r(j\omega)|^2$$
$$= |S_{21}V_1^i(j\omega)|^2$$
$$=$$

$2V_2^i = V_2 + I_2 = 0$; $I_2 = -V_2 = -V_2$

$V_2 = V_2^r$

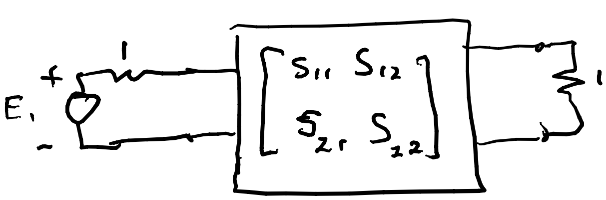
$2V_2^r = V_2 - I_2 = 2V_2$

$$S_{21}(s) = V_2^r / V_1^i \Big|_{V_2^i = 0} \Rightarrow V_2^r = S_{21}V_1^i$$

$\therefore$ $\frac{\text{Power to load}}{\text{Power available from source}}(j\omega) = \dfrac{|S_{21}(j\omega)V_1^i(j\omega)|^2}{\left.\frac{R_1}{(R_1+R_1)^2}E^2\right\} = \frac{1}{(2)^2}\cdot(2V_1^i)^2} = \dfrac{|S_{21}(j\omega)|^2|V_1^i|^2}{(V_1^i)^2}$

$$= |S_{21}(j\omega)|^2 = \text{insertion power gain}$$

$$= S_{21}(-j\omega)S_{21}(j\omega) = S_{21}(-s)S_{21}(s)\Big|_{s=j\omega}$$



$$; \quad \Sigma = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}$$

if lossless

$$\Sigma^T(-s)\Sigma(s) = 1_m$$

$$\shortparallel$$

here 1_2

$$\begin{bmatrix} S_{11}^T(-s) & S_{21}^T(-s) \\ S_{12}^T(-s) & S_{22}^T(-s)\end{bmatrix}\begin{bmatrix} S_{11}(s) & S_{12}(s)\\ S_{21}(s) & S_{22}(s)\end{bmatrix} = \begin{bmatrix}1&0\\0&1\end{bmatrix} \Rightarrow \Sigma^T(-s)\Sigma(s)=1_n$$

$1 = S_{11}^T(-s)S_{11}(s)+S_{21}^T(-s)S_{21}(s) \Rightarrow$ $S_{11}^T(-s)S_{11}(s) = 1 - S_{21}^T(-s)S_{21}(s)$

can factor $1-S_{21}^T(-s)S_{21}(s)$ into $A^T(-s)A(s)$

$(1,2) = 0 = S_{11}^T(-s)S_{12}(s) + S_{21}^T(-s)S_{22}(s)$

$(2,1) = 0 = S_{12}^T(-s)S_{11}(s)+S_{22}^T(-s)S_{21}(s)$ desire still S_{12} & S_{22} so

$(2,2) = 1 = S_{12}^T(-s)S_{12}(s)+S_{22}^T(-s)S_{22}(s)$

try $\Sigma(s)\Sigma^T(-s) = 1_n \Rightarrow 1 = S_{11}(s)S_{11}^T(-s) + S_{12}(s)S_{12}^T(-s)$

$\Rightarrow S_{12}(s)S_{12}^T(-s) = 1 - S_{11}(s)S_{11}^T(-s)$

gives S_{12} as know S_{21} which gave S_{11}

then $S_{22}^T(-s)S_{22}(s) = 1 - S_{12}^T(-s)S_{12}(s) \Rightarrow S_{22}$ by factoring $1-S_{12}^T(-s)S_{12}(s)$

given a lossless 2-port
(and the transpose is not needed here as S_{12} is a 1×1 matrix)

added example 10/28/04

Let $|S_{21}(j\omega)|^2 = \dfrac{1/4}{\omega^2+1} = S_{21}(-s)S_{21}(s)\Big|_{s=j\omega} \Rightarrow S_{21} = \dfrac{\pm 1/2}{s+1}$

choose $S_{21}(s) = +\dfrac{1/2}{s+1}$ (the choice $\dfrac{-1/2}{s+1}$ is also valid)

$1 - S_{21}(-s)S_{21}(s) = 1 - \dfrac{1/4}{1-s^2} = \dfrac{\frac{3}{4}-s^2}{1-s^2} = S_{11}(-s)S_{11}(s) = \dfrac{\left(\pm\frac{\sqrt3}{2}-s\right)\left(\pm\frac{\sqrt3}{2}+s\right)}{(1-s)(1+s)}$

factor to $S_{11}(s) = \dfrac{\frac{\sqrt3}{2}+s}{1+s}$ (the choices $\pm\frac{\sqrt3}{2}-s$ are also possible but not $(1-s)$ in denominator)

also $S_{12}(s)S_{12}(-s) = 1 - S_{11}(s)S_{11}(-s)$ ($=$ here $S_{21}(-s)S_{12}(s)$ since S_{11} is 1×1)

so we can choose $S_{12}(s) = S_{21}(s)$ or $S_{12}(s) = A(s)S_{21}(s)$ with $A(s)$ all-pass including $A(s) = -1$

Choose $S_{12}(s) = S_{21}(s)$ [guarantees a symmetric $\Sigma(s)$ that is, a reciprocal 2-port]
$= \dfrac{1/2}{s+1}$

Next find $S_{22}(s)$ from $S_{22}(-s)S_{22}(s) = 1 - S_{12}(-s)S_{12}(-s) = \dfrac{\frac{3}{4}-s^2}{1-s^2} = S_{11*}S_{11}$

choose $S_{22}(s) = -S_{11}(s) = -\left(\dfrac{\frac{\sqrt{3}}{2}+s}{1+s}\right)$ (can also insert an all-pass in the factor but that can increase the degree)

Finally we have the lossless coupling 2-port described by

$$\Sigma(s) = \begin{bmatrix} \dfrac{s+\sqrt{3}/2}{s+1} & \dfrac{1/2}{s+1} \\[2ex] \dfrac{1/2}{s+1} & -\left(\dfrac{s+\sqrt{3}/2}{s+1}\right) \end{bmatrix}$$

(this should be bounded real and satisfy $\Sigma^T(-s)\Sigma(s) = 1_2$ as the lossless condition)

If we synthesize this 2-port $S(s) = \Sigma(s)$ then we obtain the insertion power "gain" $G(\omega) = |S_{21}(j\omega)|^2 = \dfrac{1/4}{\omega^2+1}$