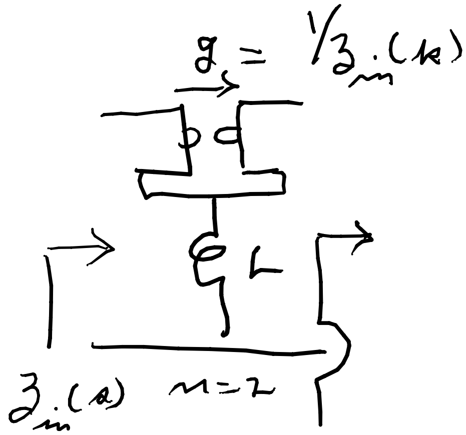
$$Z_\ell(s) = Z_{in}(k)\left[\frac{\left(\frac{s}{k}\right)\left(\frac{Z_{in}(s)}{Z_{in}(k)}\right) - 1}{\left(\frac{s}{k}\right) - \left(\frac{Z_{in}(s)}{Z_{in}(k)}\right)}\right]$$



if $Z_{in}(k) + Z_{in}(-k) = 2\,\mathrm{Ev}\,Z_{in}(s)\Big|_{s=k}$

$L = \dfrac{Z_{in}(k)}{k}$ $\quad Z_\ell(s)$

then $\delta[Z_\ell(s)] = \delta[Z_{in}(s)] - 1$

If $k>0$ then if $Z_{in}(s)$ is PR then $Z_\ell(s)$ is PR

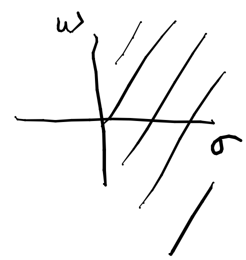
PR = positive real

define PR like p.338

PR → can make a circuit with its input impedance $Z_{in}(s)$ by using only passive elements

Positive real, $\mathbf{Z}(s)$ a square $n \times n$ matrix

1. $Z(\sigma)$ is real $\sigma > 0$ (real elements) $\quad \sigma = \mathrm{Re}\, s$
2. $Z(s)$ is analytic in $\sigma > 0$ (stable)



3. $Z(s) + Z^{T*}(s) \geq 0$ in $\sigma > 0$ (passivity)

(Hermitian part) positive semi-definite

$$P(j\omega) = \mathrm{Re}\left(V^{T*} I\right) = \frac{V^{T*} I + I^{T*} V}{2}$$

$$= \frac{1}{2}\left[I^{T*} Z^{T*}(j\omega) I + I^{T*} Z(j\omega) I\right]$$

$$= \frac{1}{2} I^{T*}\left[Z(j\omega) + Z^{T*}(j\omega)\right] I \geq 0 \text{ if passive for all } I$$

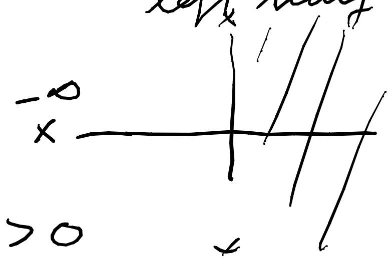
For the above 2-port

$$Z(s) = \begin{bmatrix} sL & sL-r \\ sL+r & sL \end{bmatrix} \text{ is PR}$$

1. $Z(\sigma)$ is real $\quad Z(\sigma) = \begin{bmatrix} L\sigma & \sigma L - r \\ \sigma L + r & \sigma L \end{bmatrix}$ is real if L & r are real

2. $Z(s)$ has poles at $s = \infty$ (take on $j\omega$ axis or left half plane)

$$Z(\infty) \to s\begin{bmatrix} L & L \\ L & L \end{bmatrix}$$



no poles in $\sigma > 0$

3) $Z(s) + Z^{T*}(s)$ in $\mathrm{Re}\, s > 0$

$s = \sigma + j\omega$
$\sigma > 0$

$$\begin{bmatrix} sL & sL-r \\ sL+r & sL \end{bmatrix} + \begin{bmatrix} s^*L & s^*L+r \\ s^*L-r & s^*L \end{bmatrix}$$

$$= (s+s^*)L\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = 2\sigma L\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = 2\sigma L\begin{bmatrix} 1 \\ 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \end{bmatrix}$$

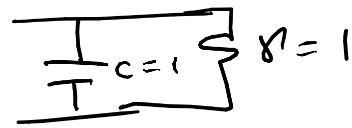
$$x^{T*}\left[Z(s) + Z^{T*}(s)\right]x = 2\sigma L\begin{bmatrix} x_1^* & x_2^* \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

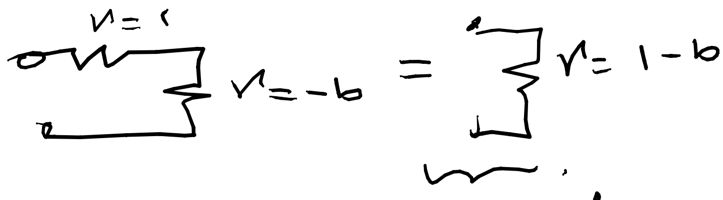
$$= 2\sigma L\left[x_1 + x_2\right]^*\left[x_1 + x_2\right] = 2\sigma L\,|x_1 + x_2|^2 \geq 0 \text{ for all } x_1 \,\&\, x_2$$

(complex x_1 & x_2), $L>0$, $\sigma>0$

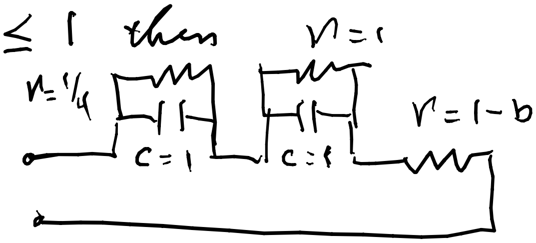
Ex: $z(s) = \underbrace{\frac{1}{s+4}}_{z_1} + \underbrace{\frac{1}{s+1}}_{z_2} + \underbrace{1-b}_{z_3}$ for what b is this PR

$\frac{1}{y_1} = z_1 \rightarrow y_1 = s+4 \Rightarrow$ 

$y_2 = \frac{1}{z_2} = s+1 \Rightarrow$ 

$z_3 = 1-b \Rightarrow$ 

If $b \le 1$ then



passive if $b<1$

Lets check $z+z^*$ in $\sigma>0$

$$z+z^*\Big|_{s=\sigma} = \frac{1}{\sigma+4} + \frac{1}{\sigma+1} - b + 1$$

for $\sigma>0$, σ very large then $z+z^* \rightarrow 1-b$

$\Rightarrow 1-b>0$ for z to be PR

$\therefore z(s) = \frac{1}{s+4} + \frac{1}{s+1} + 1 - b$ is PR iff $b \le 1$ & then we can synthesize by a passive circuit

see p. 338 to "start" this

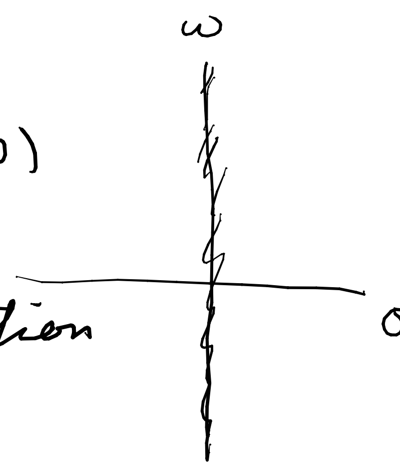
Lossless circuits $P_{ave}(j\omega) \equiv 0$ for any input I

$$I^{T*}\left[Z(j\omega) + Z^{T*}(j\omega)\right] I = 0$$

$$\therefore \quad Z(j\omega) = -Z^{T*}(j\omega)$$

if Z is real-rational $\quad = -Z^{T}(-j\omega)$

$$\Rightarrow \quad Z(j\omega) + Z^{T}(-j\omega) = 0_{n\times n}$$

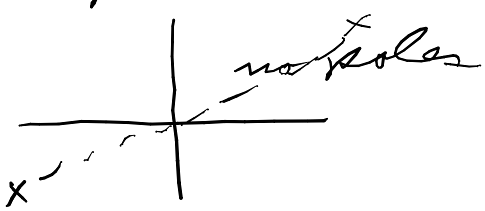


let $\omega = s/j$ $\Rightarrow$ analytic continuation

$$Z(s) + Z^{T}(-s) \equiv 0_{n\times n}$$

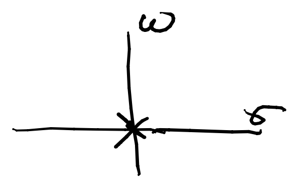
$$Z(s) = -Z^{T}(-s) \quad \text{in the whole } s \text{ plane}$$

if PR & rational $Z(s)$ has no poles in $\sigma > 0$ as $Z(s)$ is analytic there



If a pole in $\sigma < 0$ then it would be in the right half plane & this is not possible in a PR matrix $\Rightarrow$ all poles of a lossless (& PR) $Z(s)$ are on $j\omega$ axis

Ex: $z(s) = \dfrac{1}{s}$



$$z(s) + z(-s) = \frac{1}{s} + \frac{1}{-s} \equiv 0 = 2\,Ev(z(s))$$

$$2\,Re(z(s)) = \frac{1}{\sigma + j\omega} + \frac{1}{\sigma - j\omega} = \frac{2\sigma}{(\sigma^2 + \omega^2)} > 0 \text{ in } \sigma > 0$$

Result is for a scalar: If lossless all poles are on $j\omega$ axis

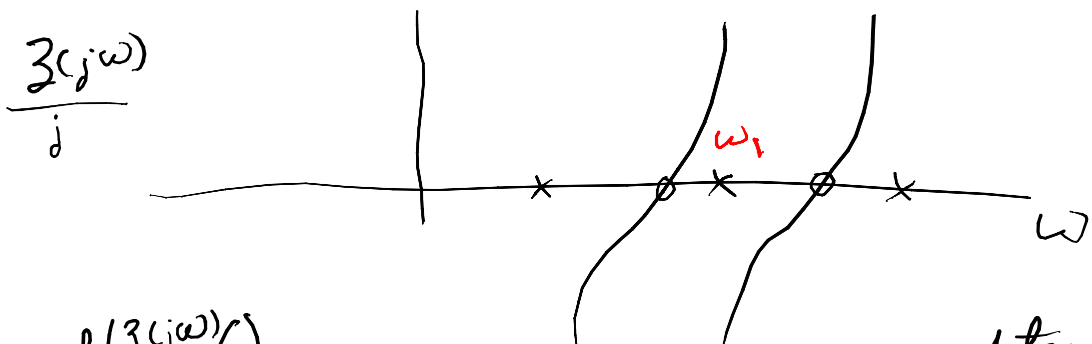
poles on the $j\omega$ are simple
& residues are real and positive

$$Z(s) = \frac{k_0}{s} + k_\infty s + \underbrace{\frac{k}{s+j\omega_1} + \frac{k}{s-j\omega_1}}_{\frac{2ks}{s^2+\omega_1^2}} + \cdots + \sum \frac{2k_i s}{s^2+\omega_i^2}$$

does have $Z(s) = -Z(-s)$
i.e. this is an odd function

$\frac{1}{Z(s)} = y(s)$ is PR if $Z(s)$ is PR; $y(s) = -y(-s)$ if lossless

there poles and zeros of $Z(s)$ lie on $j\omega$ axis and are all simple (& they alternate



if $\frac{d\left(Z(j\omega)/j\right)}{d\omega} > 0$ then poles & zeros alternate

but this is true

$$\frac{Z(j\omega)}{j} = \frac{k_0}{-\omega} + k_\infty \omega + \frac{+k(2\omega)}{-\omega^2+\omega_1^2} + \cdots$$

$$\frac{dZ(j\omega)/j}{d\omega} = \frac{k_0}{\omega^2} + k_\infty + \underbrace{\left[\frac{2k}{-\omega^2+\omega_1^2} - \frac{2k\omega(-2\omega)}{(-\omega^2+\omega_1^2)^2}\right]} + \cdots$$

$$+\,2k\left[\frac{\omega^2 + 2\omega^2 + \omega_1^2}{(-\omega^2 + \omega_1^2)^2}\right] > 0$$

≥ 0

Ex:

$$Z(s) = \frac{s(s^2+2)}{(s^2+1)}$$ this is lossless

$$= s + \frac{2ks}{s^2+1} = \frac{s^3 + s + 2ks}{s^2+1} \Rightarrow \begin{aligned} 2k+1 &= 2 \\ k &= 1/2 \end{aligned}$$

$$= s + \frac{s}{s^2+1}$$