File: c:\temp\courses\fall2003\610\vanderPol1.doc RWN 10/21/03

Realization of the van der Pol equation as an oscillator using Spice

The van der Pol equation of interest is

$$\frac{d^2x}{dt^2}+\varepsilon(x^2-1)\frac{dx}{dt}+\omega_o^2 x=0; \qquad x(0) \text{ and } \frac{dx}{dt}(0) \text{ given}$$

where $\epsilon \geqslant 0$ and $\omega_o > 0$ are constants, determining the damping and "undamped" natural frequency, respectively. Note that when x>1 there is positive "resistance" damping the oscillator while when x<1 there is negative "resistance" allowing recovery from the damping. The system has one limit cycle determined solely by these two constants and independently of the initial conditions and as such is said to be structurally stable.

We can proceed to make this with circuits to simulate in Spice by setting up state equations. Let

$$\frac{dy}{dt}=-\omega_o^2 x$$

then $\frac{d^2x}{dt^2}+\varepsilon(x^2-1)\frac{dx}{dt}=\frac{dy}{dt}$ and on integrating we have

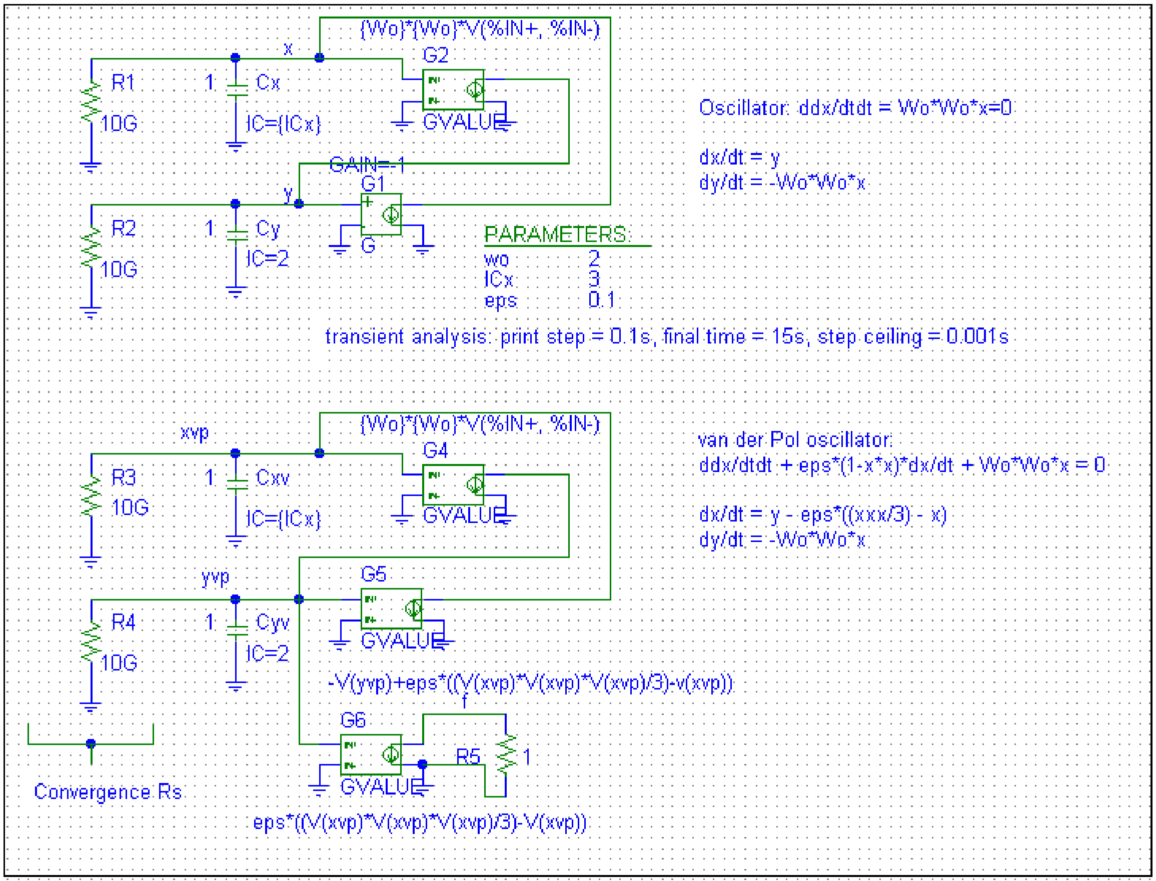
$$\frac{dx}{dt}=y+\varepsilon(x-\frac{x^3}{3})$$

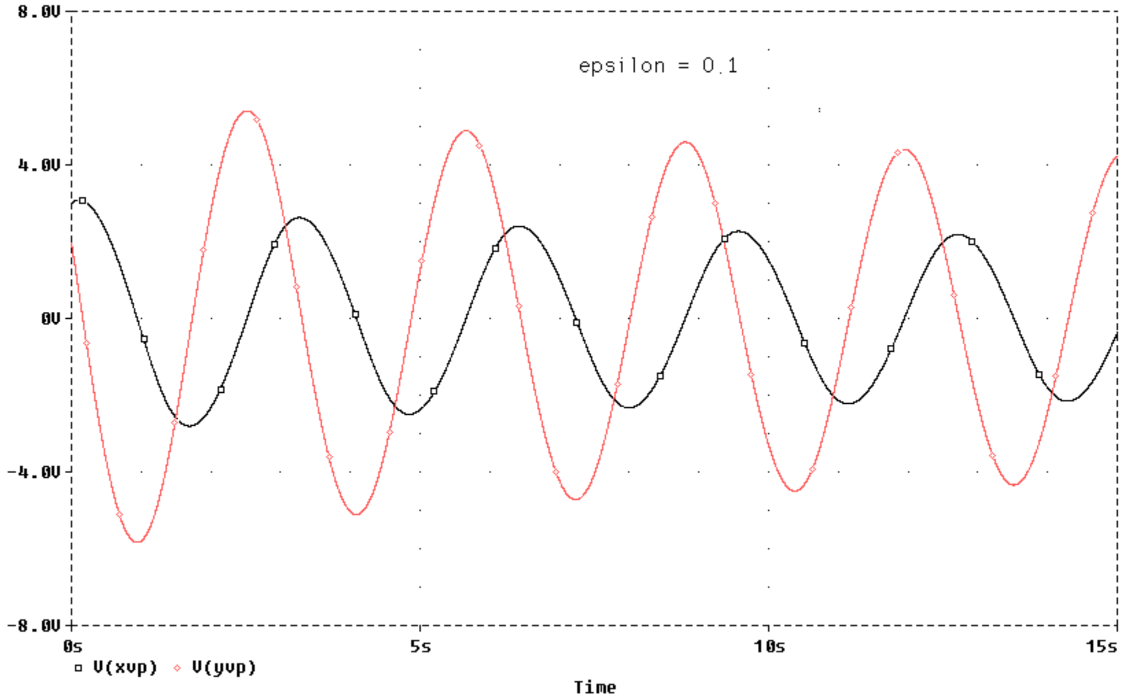
Thus we have the state variable equations

$$\frac{dx}{dt}=y+\varepsilon(x-\frac{x^3}{3}) \quad \text{given } x(0)$$

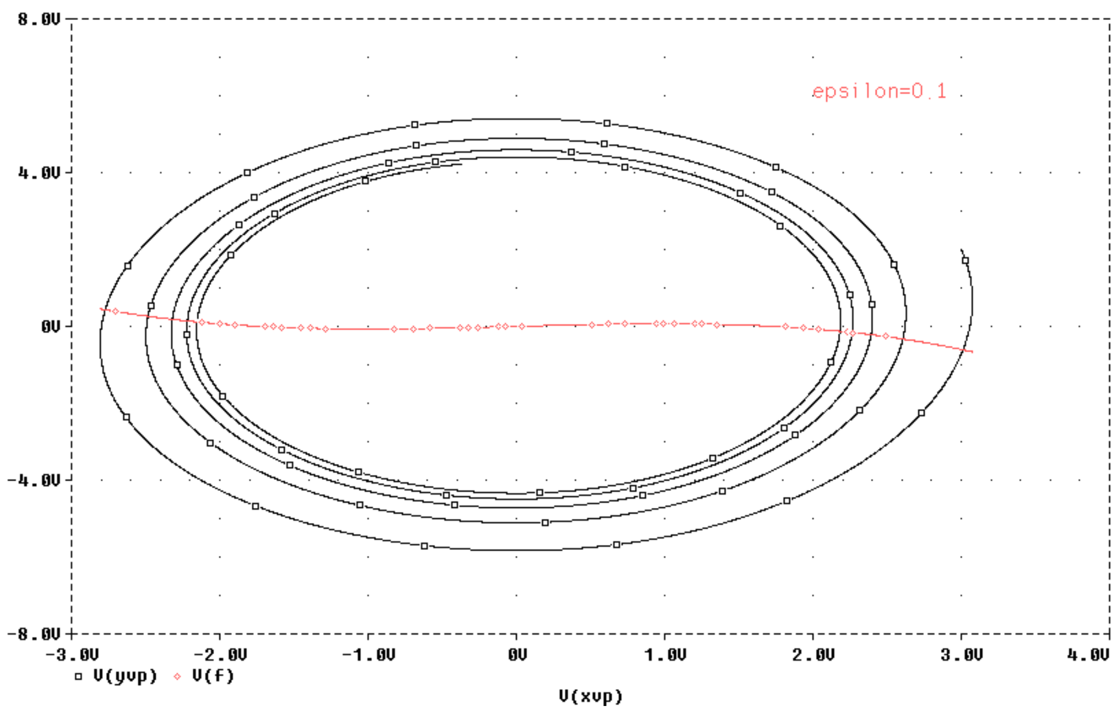
$$\frac{dy}{dt}=-\omega_o^2 x \quad \text{given } y(0)=\frac{dx}{dt}(0)-\varepsilon(x(0)-\frac{x(0)^3}{3})$$

These can be set up in Spice by making the analogy that each derivative comes from a unit capacitor as its current by differentiating the capacitor voltage, taken here to be x and y. In this case the right hand sides of the equations are to be taken as current sources controlled by the voltages. A PSpice schematic follows, where the upper circuit is for the normal undamped oscillator (the van der Pol case with ϵ=0) which is not structurally stable. The bottom circuit also includes an addition at the very bottom so that the nonlinearity can be plotted on the same graph. Below the circuit are two plots, versus time and versus the x variable, for each of two different values of ϵ to indicate the relaxation nature when ϵ is large. The state space plots contain a plot of the nonlinearity.

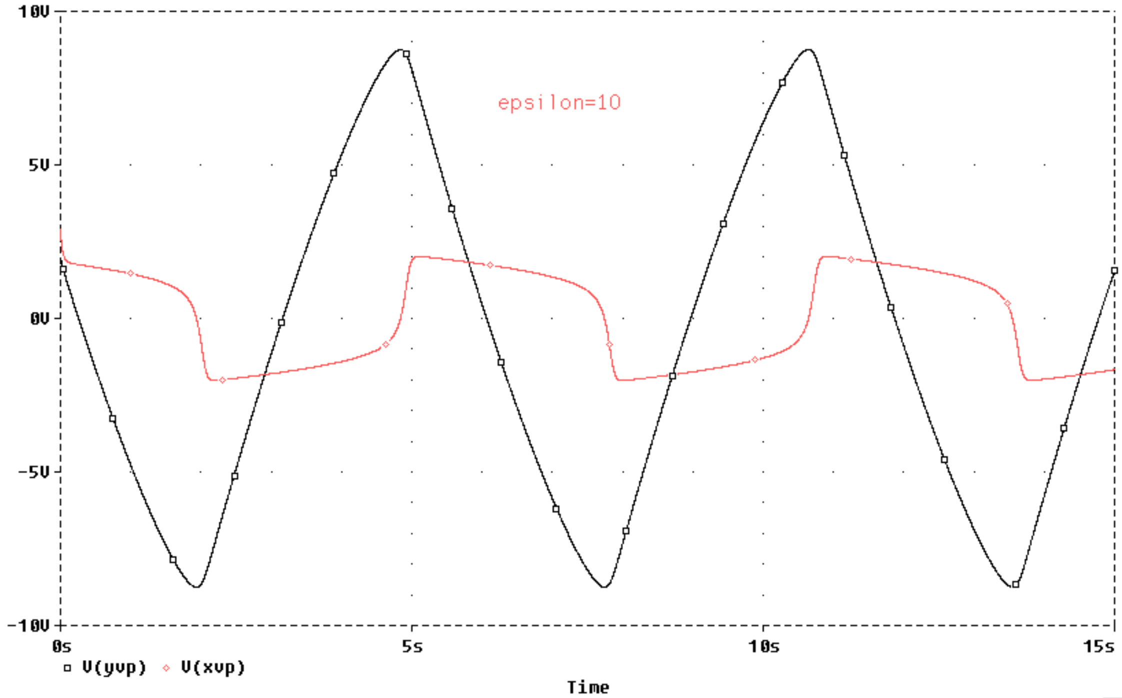

{Wo}*{Wo}*V(%IN+, %IN-)
x
G2
R1
10G
1
Cx
IC={ICx}
GVALUE
GAIN=-1
G1
y
R2
10G
1
Cy
IC=2
G
Oscillator: ddx/dtdt = Wo*Wo*x=0
dx/dt = y
dy/dt = -Wo*Wo*x
PARAMETERS:
wo 2
ICx 3
eps 0.1
transient analysis: print step = 0.1s, final time = 15s, step ceiling = 0.001s
{Wo}*{Wo}*V(%IN+, %IN-)
xvp
G4
R3
10G
1
Cxv
IC={ICx}
GVALUE
van der Pol oscillator:
ddx/dtdt + eps*(1-x*x)*dx/dt + Wo*Wo*x = 0
dx/dt = y - eps*((xxx/3) - x)
dy/dt = -Wo*Wo*x
yvp
G5
R4
10G
1
Cyv
IC=2
GVALUE
-V(yvp)+eps*((V(xvp)*V(xvp)*V(xvp)/3)-v(xvp))
f
G6
R5
1
GVALUE
Convergence Rs
eps*((V(xvp)*V(xvp)*V(xvp)/3)-V(xvp))

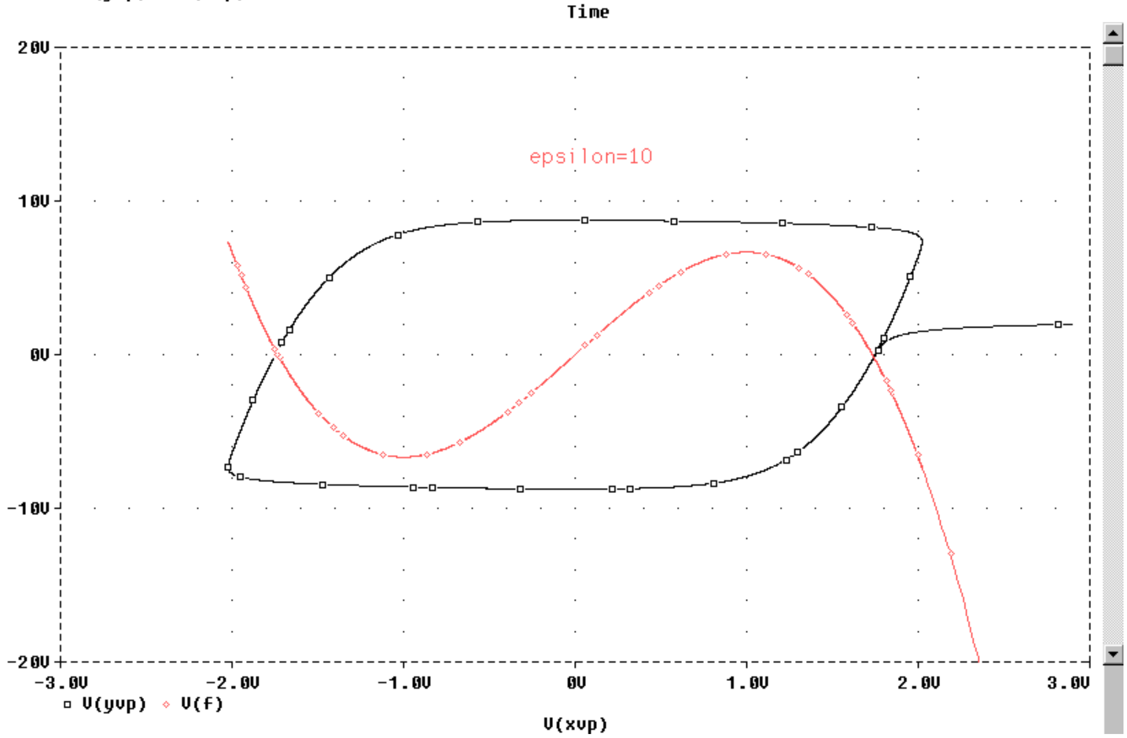


8.0V
4.0V
0V
-4.0V
-8.0V
epsilon = 0.1
0s
5s
10s
15s
V(xvp) V(yvp)
Time



8.0V
4.0V
0V
-4.0V
-8.0V
epsilon=0.1
-3.0V
-2.0V
-1.0V
0V
1.0V
2.0V
3.0V
4.0V
V(yvp) V(f)
V(xvp)


10V
5V
0V
-5V
-10V
0s
5s
10s
15s
epsilon=10
V(yvp)
V(xvp)
Time


20V
10V
0V
-10V
-20V
-3.0V
-2.0V
-1.0V
0V
1.0V
2.0V
3.0V
epsilon=10
V(yvp)
V(f)
V(xvp)