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WORST CASE LENGTH OF NEAREST NEIGHBOR TOURS FOR THE EUCLIDEAN TRAVELING SALESMAN PROBLEM*

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Abstract. The worst case length of a tour for the Euclidean traveling salesman problem produced by the nearest neighbor (NN) heuristic is studied in this paper. Nearest neighbor tours for a set of arbitrarily located points in the d-dimensional unit cube are considered. A technique is developed for bounding the worst case length of a tour. It is based on identifying sequences of *coverings* of $[0,1]^d$. Each covering $\mathcal{P}_k$ consists of sets C_i, with diameter bounded by the *diameter* $D(\mathcal{P}_k)$ of the covering. For every sequence of coverings a bound is obtained that depends on the cardinality of the coverings and their diameters. The task of bounding the worst case length of an NN tour is reduced to finding appropriate sequences of coverings. Using coverings produced by the rectangular lattice with appropriately shrinking diameter, it is shown that the worst case length of an NN tour through N points in $[0,1]^d$ is bounded by $[d\sqrt{d}/(d-1)]N^{(d-1)/d} + o(N^{(d-1)/d})$. For the unit square the tighter bound $2.482\sqrt{N} + o(\sqrt{N})$ is obtained using regular hexagonal lattice coverings.

Key words. Euclidean traveling salesman problem, nearest neighbor tours, worst case analysis

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1. Introduction.

Consider a set $V = \{x_1, \ldots, x_N\}$ of points in $[0,1]^d$. Let $G = (V, E)$ be the complete graph with vertex set V. The length of edge (x_k, x_l) is the Euclidean distance $|x_k - x_l|$ between x_k and x_l. Let $T(V)$ be the set of the tours for graph G. The tours of G are in one-to-one correspondence with the permutations of the vertices. A tour $x_{i_1}, x_{i_2}, \ldots, x_{i_N}$ with starting point x_{i_1} will be denoted by $(i_1, i_2, \ldots, i_N)$. The length of tour $t = (i_1, i_2, \ldots, i_N)$ is equal to the sum of the lengths of the edges of the tour; that is,

$$L(t) = \sum_{k=1}^{N} |x_{i_k} - x_{i_{k+1}}|,$$

where by convention $x_{i_{N+1}} = x_{i_1}$. The Euclidean traveling salesman problem (TSP) is to find the minimum length tour through the set of points V.

The TSP is one of the most heavily studied problems of combinatorial optimization [4, 8]. In general graphs where the length of the edges may be arbitrary the TSP was among the first problems shown to be NP-complete (see Karp [7]). The Euclidean TSP also has been shown to be NP-complete (see Papadimitriou [10]). There has been a lot of work on heuristics and approximate algorithms with guaranteed performance [5].

A popular heuristic for the Euclidean TSP is the nearest neighbor (NN) algorithm. According to this the tour is derived by selecting an arbitrary initial point x_{i_1} and visiting successively from point x_{i_k} the point $x_{i_{k+1}}$, which is the closest to x_{i_k}, among

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those that have not been visited yet. Hence any tour $t = (i_1, i_2, \ldots, i_N)$ produced by the NN heuristic satisfies the property

$$|x_{i_{k+1}} - x_{i_k}| = \min_{j=k+1,\ldots,N} |x_{i_k} - x_{i_j}|, \; k = 1, \ldots, N-1. \tag{1}$$

Also, any tour satisfying (1) can be produced by the NN heuristic if the starting point is selected accordingly. Any tour that satisfies property (1) will be called an NN tour throughout the rest of the paper. Denote by $NN(V)$ the set of all NN tours that correspond to the set of points V. The objective of this paper is to study the worst case length of an NN tour over all configurations $V = \{x_1, \ldots, x_N\}$ of N points in the d-dimensional unit cube $[0,1]^d$. This is defined as

$$L_N = \sup_{V \subset [0,1]^d, |V| = N} \max_{t \in NN(V)} L(t).$$

One way to assess the performance of the NN heuristic is to compare L_N with the length of the worst case minimum length tour

$$P_N = \sup_{V \subset [0,1]^d, |V| = N} \min_{t \in T(V)} L(t).$$

There are several studies on obtaining bounds for P_N. Steele [11] contains a detailed account of related results. Few [3] obtained an upper bound on P_N for the general d-dimensional case; that is,

$$P_N \leq d\{2(d-1)\}^{(1-d)/2d} N^{(d-1)/d} + o(N^{1-2/d}).$$

This was further improved for large d by Moran [9], while Karloff [6] improved the upper bound for $d = 2$ by showing that $P_N \leq 0.984\sqrt{2}\sqrt{N} + c$. For the two-dimensional case, Supowit, Reingold, and Plaisted [12] proved that

$$\left(\frac{4}{3}\right)^{1/4} \sqrt{N} - o(\sqrt{N}) \leq P_N.$$

The performance of the NN heuristic has been previously studied for the TSP in general graphs as well as in more special cases of graphs that satisfy certain constraints. Johnson and Papadimitriou [5] review related work.

Upper bounds on L_N are obtained in this paper for the Euclidean TSP. From these bounds and well-known lower bounds on P_N it follows that for the Euclidean TSP in the unit square, the ratio L_N/P_N is bounded asymptotically by 2.3095 or, more precisely, that for every $\epsilon > 0$, there exists $N(\epsilon)$ such that $L_N/P_N \leq 2.3095 + \epsilon$ for $N > N(\epsilon)$. Similar results follow for NN tours in higher dimensions from the corresponding bounds on L_N in higher dimensions.

The rest of the paper is organized as follows. In section 2 the technique for bounding L_N using coverings of $[0,1]^d$ is presented and the main bounding theorem is obtained. In section 3 this technique is applied with coverings derived from the regular rectangular lattice and a bound for NN tours in d dimensions is obtained. In section 4 the bound is tightened for the unit square using regular hexagonal coverings. Some further points are discussed in section 5.

2. The general bound. The diameter $D(C)$ of a subset C of R^d is defined as

$$D(C) = \sup_{x,y \in C} |x - y|.$$

The essential property of an NN tour used in the derivation of the bounds in this paper is captured in the following lemma.

LEMMA 2.1. *For any subset C of R^d and tour $t = (i_1, i_2, \ldots, i_N)$ produced by the NN heuristic, there is at most one vertex $x_{i_k} \in C$ such that*

$$|x_{i_k} - x_{i_{k+1}}| > D(C).$$

Proof. By contradiction, assume there are two vertices x_{i_l}, x_{i_m} in C such that $|x_{i_l} - x_{i_{l+1}}| > D(C)$ and $|x_{i_m} - x_{i_{m+1}}| > D(C)$. Without loss of generality assume that $l < m$. Then

$$|x_{i_m} - x_{i_l}| \geq \min_{j=l+1,\ldots,n} |x_{i_j} - x_{i_l}| = |x_{i_{l+1}} - x_{i_l}| = |x_{i_l} - x_{i_{l+1}}| > D(C) \geq |x_{i_m} - x_{i_l}|,$$

which contradicts from (1) the assumption that t is an NN tour. □

Lemma 2.1 will be used to derive the main bounding theorem in the following, after some preliminary definitions.

A *covering* $\mathcal{P}$ of a set $A \subset R^d$ is defined to be any collection of subsets of R^d, $\mathcal{P} = \{C_l : l = 1, \ldots, P\}$, $C_l \subset R^d$, $l = 1, \ldots, P$, with the property $\cup_{l=1}^{P} C_l \supseteq A$. The sets that constitute the covering will be called *cells* of the covering in the following. The diameter $D(\mathcal{P})$ of the covering $\mathcal{P}$ is defined as

$$D(\mathcal{P}) = \max_{l=1,\ldots,P} D(C_l).$$

The cardinality P of covering $\mathcal{P}$ will be denoted as $|\mathcal{P}|$.

Note that for every covering, a bound on an NN tour can be obtained easily using Lemma 2.1. In any cell there can be at most one point with an adjacent edge of the tour that has length greater than the cell diameter. Hence, at most $|\mathcal{P}|$ edges of an NN tour will have length larger than the diameter of the covering, while the length of all the other edges will be smaller than $D(\mathcal{P})$. Therefore,

$$L_N \leq (N - |\mathcal{P}|)D(\mathcal{P}) + |\mathcal{P}|D(A), \tag{2}$$

where the fact that the length of any edge of an NN tour will be less than $D(A)$ has been used. By considering sequences of coverings instead of a single covering, bounds tighter than (2) can be obtained. In the rest of the paper by "covering" we will mean the covering of $[0, 1]^d$.

Consider sequences of coverings $\mathcal{P}_m$, $m = 1, \ldots, M$ with decreasing diameter, where

$$D(\mathcal{P}_m) \geq D(\mathcal{P}_{m+1}), \; m = 1, \ldots, M - 1.$$

The following theorem holds.

THEOREM 2.2. *The worst case length of an NN tour is bounded as follows:*

$$L_N \leq ND(\mathcal{P}_M) + \sum_{m=2}^{M} |\mathcal{P}_m|(D(\mathcal{P}_{m-1}) - D(\mathcal{P}_m)) + |\mathcal{P}_1|(D(A) - D(\mathcal{P}_1)). \tag{3}$$

Proof. It is shown that for an arbitrary tour $t = (i_1, \ldots, i_N)$,

$$(4) \quad L(t) \le ND(\mathcal{P}_M) + \sum_{m=2}^{M} |\mathcal{P}_m|(D(\mathcal{P}_{m-1}) - D(\mathcal{P}_m)) + |\mathcal{P}_1|(D(A) - D(\mathcal{P}_1)).$$

Consider the increasing sequence of subsets of vertices V_m, $m = 1, \ldots, M$ defined as follows:

$$V_m = \{x_{i_k} : x_{i_k} \in V, |x_{i_k} - x_{i_{k+1}}| > D(\mathcal{P}_m)\}.$$

Note that the sets $V - V_M$, $V_M - V_{M-1}$, $V_{M-1} - V_{M-2}, \ldots, V_2 - V_1, V_1$ constitute a partition of V. Therefore, the length of tour t can be written as follows:

$$(5) \quad L(t) = \sum_{x_{i_k} \in (V - V_M)} |x_{i_k} - x_{i_{k+1}}| + \sum_{m=2}^{M} \sum_{x_{i_k} \in (V_m - V_{m-1})} |x_{i_k} - x_{i_{k+1}}| + \sum_{x_{i_k} \in V_1} |x_{i_k} - x_{i_{k+1}}|.$$

By the definition of the sets V_i,

$$(6) \quad |x_{i_k} - x_{i_{k+1}}| \le D(\mathcal{P}_M), \; x_{i_k} \in (V - V_M),$$

$$(7) \quad |x_{i_k} - x_{i_{k+1}}| \le D(\mathcal{P}_{m-1}), \; x_{i_k} \in (V_m - V_{m-1}), \; m = 2, 3, \ldots, M,$$

$$(8) \quad |x_{i_k} - x_{i_{k+1}}| \le D(A), \; x_{i_k} \in V_1.$$

By substituting from equations (6), (7), and (8) to equation (5), we get

$$(9) \quad L(t) \le |V - V_M| D(\mathcal{P}_M) + \sum_{m=2}^{M} |V_m - V_{m-1}| D(\mathcal{P}_{m-1}) + |V_1| D(A).$$

Since $V_1, V_2, \ldots, V_M, V$ is an increasing sequence of sets ($V_m \subseteq V_{m+1}$), we have $|V_m - V_{m-1}| = |V_m| - |V_{m-1}|$, $m = 2, \ldots, M$, and substituting in (9) we get

$$(10) \quad L(t) \le (|V| - |V_M|) D(\mathcal{P}_M) + \sum_{m=2}^{M} (|V_m| - |V_{m-1}|) D(\mathcal{P}_{m-1}) + |V_1| D(A).$$

By rearranging the sum in the right-hand side of (10), we get

$$(11) \quad L(t) \le |V| D(\mathcal{P}_M) + \sum_{m=2}^{M} |V_m|(D(\mathcal{P}_{m-1}) - D(\mathcal{P}_m)) + |V_1|(D(A) - D(\mathcal{P}_1)).$$

Note that relationship (11) holds for any TSP tour. The fact that t is an NN tour is now used to bound $|V_m|$. From Lemma 2.1 we have that any cell C of covering $\mathcal{P}_m$ can contain at most one point x_{i_k}, such that $|x_{i_k} - x_{i_{k+1}}| > D(C)$. Therefore, each cell of $\mathcal{P}_m$ can contribute at most one point to the set V_m; hence

$$(12) \quad |V_m| \le |\mathcal{P}_m|, \; m = 1, \ldots, M.$$

Substituting in inequality (11) from (12) we get (4). □

In the next two sections it is shown how Theorem 2.2 can be applied to specific coverings to get bounds on L_N.

3. Bounds from rectangular lattice coverings. In this section a bound on L_N is obtained using the coverings implied by the rectangular lattice. Consider the sequence of coverings $\mathcal{P}_k,\ k = 1, \ldots, M$, where

$$\mathcal{P}_k = \{C_{l_1 l_2 \ldots l_d} : l_i = 0, 1, \ldots, k-1,\ i = 1, \ldots, d\}$$

and

$$C_{l_1 l_2 \ldots l_d} = \left\{ \left(\frac{l_1}{k} + x_1, \frac{l_2}{k} + x_2, \ldots, \frac{l_d}{k} + x_d \right) : 0 \le x_i < \frac{1}{k},\ i = 1, \ldots, d \right\}.$$

That is, the cells of the covering are d-dimensional cubes with edge length $1/k$. By applying Theorem 2.2 with the sequence of coverings above, we have the following.

THEOREM 3.1. *The worst case length of a tour through N points in $[0,1]^d$ produced by the NN heuristic is bounded as follows:*

(13)
$$L_N \le \sum_{m=1}^{d-1} \frac{(d-m+1)\sqrt{d}}{d-m} N^{(d-m)/d} + \ln(N^{1/d} - 1) + 1 + \frac{1}{N^{1/d} - 1} - \sqrt{d} \sum_{m=1}^{d-1} \frac{1}{d-m}.$$

Proof. Note that the diameter of all cells in covering $\mathcal{P}_k$ is equal to $\sqrt{d}/k$; therefore,

$$D(\mathcal{P}_k) = \frac{\sqrt{d}}{k}$$

and also

$$|\mathcal{P}_k| = P_k = k^d.$$

By applying Theorem 2.2 to this covering, we get

$$\begin{aligned} L_N &\le N\sqrt{d}\frac{1}{M} + \sum_{k=2}^{M} k^d \left(\frac{\sqrt{d}}{k-1} - \frac{\sqrt{d}}{k} \right) \\ &= N\sqrt{d}\frac{1}{M} + \sqrt{d}\sum_{k=2}^{M} \left(\frac{k^{d-2}}{k-1} + k^{d-2} \right). \end{aligned}$$

Using the formula for the sum of a geometric series, we get

$$L_N \le N\sqrt{d}\frac{1}{M} + \sqrt{d}\sum_{k=2}^{M} \left(k^{d-2} + k^{d-3} + \cdots + 1 + \frac{1}{k-1} \right). \tag{14}$$

Substituting in (14) using the well-known bounds (see [2]),

$$\sum_{k=m}^{n} f(k) \le \int_{m-1}^{n} f(x)dx$$

for the sums in the parentheses in (14), and after some calculations we get

$$L_N \le \frac{N\sqrt{d}}{M} + \sum_{m=1}^{d-1} \frac{\sqrt{d}}{d-m} M^{d-m} + \ln(M-1) - \sqrt{d} \sum_{m=1}^{d-1} \frac{1}{d-m} + 1. \tag{15}$$

Inequality (15) holds for all values of M. For $M = \lfloor N^{1/d} \rfloor$, inequality (15) becomes

$$(16)\ L_N \leq \frac{N\sqrt{d}}{\lfloor N^{1/d} \rfloor} + \sum_{m=1}^{d-1} \frac{\sqrt{d}}{d-m} \lfloor N^{1/d} \rfloor^{d-m} + \ln(\lfloor N^{1/d} \rfloor - 1) - \sqrt{d} \sum_{m=1}^{d-1} \frac{1}{d-m} + 1.$$

By replacing the floors in (16) such that the inequality remains true, we get

$$(17)\ L_N \leq \frac{N\sqrt{d}}{N^{1/d} - 1} + \sum_{m=1}^{d-1} \frac{\sqrt{d}}{d-m} N^{(d-m)/d} + \ln(N^{1/d} - 1) - \sqrt{d} \sum_{m=1}^{d-1} \frac{1}{d-m} + 1.$$

By using the formula for the sum of a geometric series in the term $N\sqrt{d}/(N^{1/d} - 1)$, we finally get

$$L_N \leq \sum_{m=1}^{d-1} \frac{(d-m+1)\sqrt{d}}{d-m} N^{(d-m)/d} + \frac{1}{N^{1/d} - 1} + \ln(N^{1/d} - 1) - \sqrt{d} \sum_{m=1}^{d-1} \frac{1}{d-m} + 1,$$

and the proof is complete. □

Note that the higher-order term of the bound in Theorem 3.1 is $[d\sqrt{d}/(d-1)]N^{(d-1)/d}$. The bound in Theorem 2.2 depends on the type of coverings used in the derivation. By selecting the appropriate type of cells in the coverings, the derived bound can be tightened, as is shown in the following for the unit square.

4. Tighter bounds for the unit square using the regular hexagonal lattice. Consider coverings of the unit square using the hexagonal lattice. The covering $\mathcal{P}_k$ consists of hexagons with diameter $2/\sqrt{3}k$, arranged as depicted in Figure 1. Hence the diameter of the covering is

$$(18)\qquad D(\mathcal{P}_k) = 2/(\sqrt{3}k).$$

By counting the cells in the covering carefully we can verify that

$$(19)\qquad |\mathcal{P}_k| \leq (2k+1)\frac{k}{\sqrt{3}} + 3k + 1.$$

Considering the sequence $\mathcal{P}_k$, $k = 1, \ldots, M$ of coverings $\mathcal{P}_k$ as above and using Theorem 2.2, we can obtain the following.

THEOREM 4.1. *The worst case length of a tour through N points in $[0,1]^2$ produced by the NN heuristic is bounded as follows:*

$$(20)\qquad L_N \leq 2^{5/2}3^{-3/4}\sqrt{N} + \frac{10\sqrt{3}+2}{3\sqrt{3}} \ln\left(\left(\frac{3}{4}\right)^{1/4}\sqrt{N}\right) + \frac{4\sqrt{3}+5}{3}.$$

Proof. Applying Theorem 2.2 with the hexagonal coverings $\mathcal{P}_k$, $k = 1, \ldots, M$ and using (18) and (19), we obtain

$$(21)\ L_N \leq \frac{2N}{\sqrt{3}M} + \sum_{m=2}^{M} \left(\frac{2}{\sqrt{3}}m^2 + \frac{1+3\sqrt{3}}{3}m + 1\right)\left(\frac{2}{\sqrt{3}(m-1)} - \frac{2}{\sqrt{3}m}\right) + 3.$$

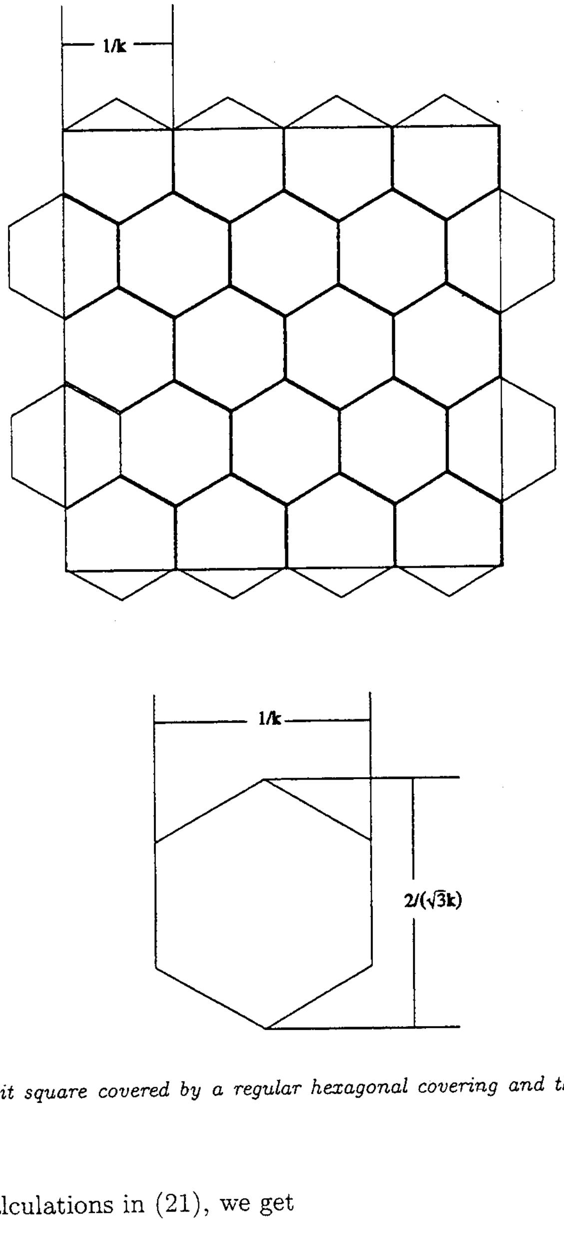

FIG. 1. *The unit square covered by a regular hexagonal covering and the cell of the covering are depicted.*

By doing some calculations in (21), we get

$$L_N \le \frac{2N}{\sqrt{3}M} + \sum_{m=2}^{M} \frac{2}{\sqrt{3}} m^2 \frac{2}{\sqrt{3}m(m-1)} + \sum_{m=2}^{M} \frac{1+3\sqrt{3}}{3} \cdot \frac{2m}{\sqrt{3}m(m-1)} + \sum_{m=2}^{M} \frac{2}{\sqrt{3}m(m-1)} + 3,$$

from which we finally get

$$L_N \le \frac{2N}{\sqrt{3}M} + \frac{4}{3}(M-2) + \sum_{m=2}^{M} \frac{10\sqrt{3}+2}{3\sqrt{3}(m-1)} + \sum_{m=2}^{M} \frac{2}{\sqrt{3}m(m-1)} + 3. \tag{22}$$

Substituting in (22) using the bounds

$$\sum_{k=m}^{n} f(k) \leq \int_{m-1}^{n} f(x)dx$$

for the summations, we get

$$L_N \leq \frac{2N}{\sqrt{3}M} + \frac{4}{3}(M-2) + \frac{10\sqrt{3}+2}{3\sqrt{3}} \ln(M-1) + \frac{4}{\sqrt{3}} + 3. \tag{23}$$

By selecting $M = \lceil (3/4)^{1/4}\sqrt{N} \rceil$, equation (23) becomes

$$L_N \leq \frac{2N}{\sqrt{3}\lceil (3/4)^{1/4}\sqrt{N} \rceil} + \frac{4}{3}(\lceil (3/4)^{1/4}\sqrt{N} \rceil - 2) + \frac{10\sqrt{3}+2}{3\sqrt{3}} \ln(\lceil (3/4)^{1/4}\sqrt{N} \rceil - 1) + \frac{4}{\sqrt{3}} + 3. \tag{24}$$

Replacing the ceilings in equation (24) such that the inequality remains true and after some calculations, we get

$$L_N \leq 2^{3/2}3^{-3/4}\sqrt{N} + \frac{4}{3}\left(\left(\frac{3}{4}\right)^{1/4}\sqrt{N} - 1\right) + \frac{10\sqrt{3}+2}{3\sqrt{3}} \ln\left(\left(\frac{3}{4}\right)^{1/4}\sqrt{N}\right) + \frac{4}{\sqrt{3}} + 3, \tag{25}$$

from which the theorem follows after simple calculations. □

Note that the highest-order term of the bound of Theorem 3.1 for the two-dimensional case is $2.84\sqrt{N}$, while the highest-order term of the bound of Theorem 4.1 is equal to $2.482\sqrt{N}$.

5. Discussion. A methodology for bounding the length of NN tours in Euclidean TSPs using coverings of $[0,1]^d$ was presented in this paper. The general bound in section 2 is proportional to both the diameter of the covering and its cardinality. Hence, in order to obtain good bounds, it is important to find coverings with small diameter and as small cardinalities as possible. In two dimensions, hexagonal coverings achieve a better trade-off between cardinality and diameter than rectangular coverings, and consequently the bound that was obtained using these coverings in section 4 is better than the one obtained by using rectangular coverings. In fact, the hexagonal covering is the one that achieves the optimal trade-off between diameter and cardinality in two dimensions, as is mentioned in the book of Conway and Sloane [1], where coverings and their properties are studied extensively.

REFERENCES

[1] J. H. Conway and N. J. A. Sloane, *Sphere Packings Lattices and Groups*, Springer-Verlag, New York, Berlin, 1993.

[2] T.H. Corman, C.E. Leiserson, and R.L. Rivest, *Introduction to Algorithms*, McGraw–Hill, New York, 1990.

[3] L. Few, *The shortest path and the shortest road through n points in a region*, Mathematika, 2 (1955), pp. 141–144.

[4] M. R. Garey and D. S. Johnson, *Computers and Intractability: A Guide to the Theory of NP-Completeness*, Freeman, San Francisco, CA, 1979.

[5] D. S. Johnson and C. H. Papadimitriou, *Performance guarantees for heuristics*, in The Traveling Salesman Problem: A Guided Tour in Combinatorial Optimization, E. L. Lawler, J. K. Lenstra, A. H. G. Rinnooy Kan, and D. B. Shmoys, eds., John Wiley, New York, 1985, pp. 145–180.

[6] H. Karloff, *How long can a Euclidean traveling salesman tour be?* SIAM J. Discrete Math., 2 (1989), pp. 91–99.
[7] R. M. Karp, *Reducibility among combinatorial problems*, in Complexity of Computer Computations, R. E. Miller and J. W. Thacher, eds., Plenum, New York, 1972, pp. 85–103.
[8] E. L. Lawler, J. K. Lenstra, A. H. G. Rinnooy Kan, and D. B. Shmoys, eds., *The Travelling Salesman Problem*, John Wiley, New York, 1985.
[9] S. Moran, *On the length of optimal TSP circuits in sets of bounded diameter*, J. Combin. Theory Ser. B, 37 (1984), pp. 113–141.
[10] C. H. Papadimitriou, *The Euclidean traveling salesman problem is NP-complete*, Theoret. Computer Sci., 4 (1977), pp. 237–244.
[11] J. M. Steele, *Probabilistic and worst case analyses of classical problems of combinatorial optimization in Euclidean space*, Math. Oper. Res., 15(4) (1990), pp. 749–770.
[12] K. J. Supowit, E. M. Reingold, and D. A. Plaisted, *The traveling salesman problem and minimum matching in the unit square*, SIAM J. Comput., 12 (1983), pp. 144–156.