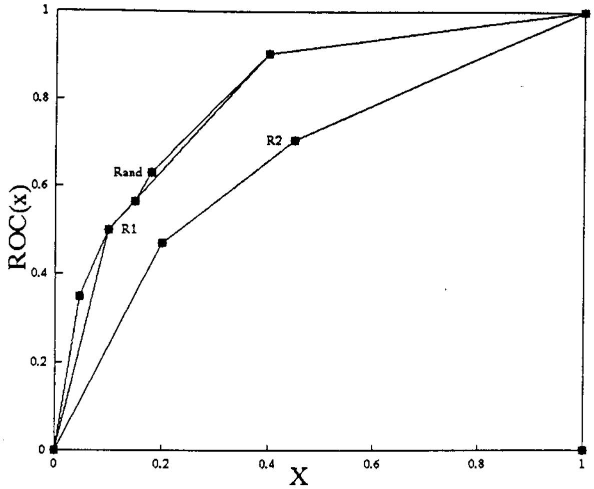

Fig. 5. Example of nonconcavity of R_0^{AND}.

Fig. 5 gives the graph of R_1, R_2, and R_0^{AND}. We see that R_0^{AND} is not a concave function.

V. Discussion and Conclusions

The obvious conclusion is that these are treacherous waters, and something may hold very often, without being true. It may even, as in the case of our first two examples, be supported by a plausible argument, but remain false.

The second example, revealing a breaking of the obvious symmetry, is less surprising, but may have more of a cautionary impact. Failure of this naive symmetry makes us more doubtful about the larger symmetry assumed in the restriction to k-out-of-n rules. That restriction has been of practical importance, in reducing astronomical numbers of possibilities to small numbers of possibilities. It leads us to suspect that the sufficiency of k-out-of-n rules, even for identical sensors, may never be established as a theorem, but will remain a heuristic. The more positive conclusion of our analysis is that the deviations found here, which show certain conjectures to be false, are all numerically small. This holds open the possibility that they are small in every case, so that assuming the conjectures to be true will lead to small numerical errors in the determination of the optimal tuning and fusion of a distributed sensor system.

Acknowledgment

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References

[1] R. R. Tenney and N. R. Sandell, Jr., "Detection with distributed sensors," *IEEE Trans. Aerospace Electron. Syst.*, vol. AES-17, pp. 501-510, July 1981.

[2] S. C. A. Thomopoulos, R. Viswanathan, and D. K. Bougolias, "Optimal data fusion in multiple sensor systems," *IEEE Trans. Aerospace Electron. Syst.*, vol. AES-23, pp. 644-653, Sept. 1986.

[3] S. C. A. Thomopoulos, R. Viswanathan, and R. Tumuluri, "Optimal serial distributed decision fusion," *IEEE Trans. Aerospace Electron. Syst.*, vol. AES-24, pp. 366-375, July 1988.

[4] S. C. A. Thomopoulos, R. Viswanathan, and D. K. Bougolias, "Optimal data distributed decision fusion," *IEEE Trans. Aerospace Electron. Syst.*, vol. AES-25, pp. 761-765, Sept. 1989.

[5] H. R. Hashemi and I. B. Rhodes, "Decentralized sequential detection," *IEEE Trans. Inform. Theory*, vol. 35, pp. 509-520, May 1989.

[6] V. V. S. Sarma and K. A. G. Rao "Decentralized detection and estimation in distributed sensor systems," in *Proc. IEEE 1983 Cybern. Society Conf.*, vol. 1, 1983, pp. 438-441.

[7] A. R. Reibman and L. W. Nolte, "Optimal detection and performance of distributed sensor systems," *IEEE Trans. Aerospace Electron. Syst.*, vol. AES-23, pp. 24-30, Jan. 1987.

[8] L. K. Ekchian, "Optimal design of distributed detection networks," unpublished Ph.D. dissert., M.I.T., Cambridge, MA, 1983.

[9] M. Cherikh, "Optimal decision and detection in the decentralized case," unpublished Ph.D. dissert., Case Western Reserve Univ., Cleveland, OH, 1989.

[10] P. B. Kantor, R. Blankenbecler, and M. Cherikh, "Sensor calculus: Considerations in the design of distributed systems for detection, discrimination and decision," Report Tantalus/CT-88/3, 1988.

[11] M. Longo, T. D. Lookabaugh, and R. M. Gray, "Quantization for decentralized hypothesis testing under communication constraints," *IEEE Trans. Inform. Theory*, vol. 36, pp. 241-255, Mar. 1990.

[12] J. Tsitsiklis, "On threshold rules in decentralized detection," in *Proc. 25th Conf. on Decision and Contr.*, 1986, pp. 232-236.

[13] P. Willett and D. Warren, "Randomization and scheduling in decentralized detection," in *Proc. 27th Ann. Conf. on Commun. Contr., and Comput.*, 1989, pp. 649-650.

Jointly Optimal Routing and Scheduling in Packet Radio Networks

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Abstract—**A multihop packet radio network is considered with a single traffic class and given end-to-end transmission requirements. A transmission schedule specifies at each time instant the set of links which are allowed to transmit. The purpose of a schedule is to prevent interference among transmissions from neighboring links. Given amounts of information are residing initially at a subset of the network nodes and must be delivered to a prespecified set of destination nodes. The transmission schedule that evacuates the network in minimum time is specified. The decomposition of the problem into a pure routing and a pure scheduling problem is crucial for the characterization of the optimal transmission schedule.**

Index Terms—**Radio networks, scheduling, routing, throughput, multiple access, delay, protocol, network topology.**

I. Introduction

In this correspondence, we study the problem of joint link activation and route selection in Packet Radio Networks (PRN's). We consider the case of network evacuation, that is the case in which we wish to deliver all packets initially residing at each node of the network to a fixed, common destination node. At each node we assume that there exists a single transceiver. Consequently, to ensure conflict-free transmissions, no two links that share a common node may be activated simultaneously. We also assume that suitable spread-spectrum signaling modulation is used, so that no additional restriction on simultaneous link activation is needed to ensure conflict-free communication, i.e., there is no "hidden terminal" problem [7]. The problem of scheduling link activation in PRN's has been studied extensively under various assumptions [1]-[3], [5], [6]. Hajek and Sasaki in [1] have studied the optimal scheduling problem for given link flow requirements. They derived an algorithm of polynomial time complexity that solves the problem of pure

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scheduling when link flows are specified. They also show that the link flows can be obtained from given end-to-end transmission demands, by means of a linear program.

In the case that we consider (namely that of evacuating a PRN from initial amounts of information in minimum time), the problems of route selection and link activation, scheduling are coupled. In Section II, we show how the joint optimization problem decomposes into a pure scheduling problem and a pure routing problem. The routing and scheduling problems are those that have been studied in [1]. We show that the danger of "under flow" in the evacuation schedule can be averted.

The computed "optimal" schedule in most cases is not practical since it requires changes of the transmission set too frequently. However, its performance, namely, the computed optimal length, is a strict lower bound for the length of any other schedule that achieves the evacuation goal and, thus, it can be used as a benchmark for comparison to the performance of heuristic schemes.

II. Problem Formulation and Decomposition

We represent the PRN with a directed graph $G = (V, E)$ where the nodes of the graph correspond to communication nodes and the edge (i, j) denotes a radio link directed from node i to node j. According to the operating assumptions that we have made in the introduction, a subset of links T can be operated simultaneously without conflicts if every two links that belong to the set do not share a common node. A subset of links with the above property is called a *transmission set*. Note that the transmission sets of the PRN are the matchings[1] of the graph G. To each transmission set T, we associate an indicator vector $I^T \in R^{|E|}$. The elements of the vector correspond to the edges of the PRN and they are given by

$$I_j^T = \begin{cases} 1, & \text{if } j \in T, \\ 0, & \text{otherwise}. \end{cases}$$

We wish to specify at each time instant the transmission set that is activated. Thus, a schedule s of link activations is a sequence of pairs each of which consists of a transmission set and the corresponding duration of its activation time; namely,

$$s = \{(T_i, \tau_i),\ i = 1, \cdots, N\},$$

where N is the total number of distinct activation epochs (note that we permit a transmission set to repeat itself in the sequence). The length of the schedule s is defined by $L(s) = \sum_{i=1}^{N} \tau_i$. Clearly set T_i is activated during the interval (t_{i-1}, t_i), where $t_i = \sum_i^{j=1} \tau_j$, and $t_0 = 0$. We consider the information residing at each node to be described by a quantity taking values in a continuum. Thus, we let $q(t) \triangleq [q_1(t), \cdots, q_{|V|}(t)]$, where $q_i(t)$ is the amount of information that resides at node i at time t. We further assume that all links have transmission capacity of one unit of information per unit of time. Thus, given an initial amount of information $q(0)$ residing at the nodes of the PRN at time $t = 0$, and assuming that no new information enters the network from exogenous sources, we can obtain the amount of information on the nodes at time t for a given schedule s by means of the following[2]

$$q(t) = \max\left(q(t_{k-1}) - (t - t_{k-1}) A I^{T_k}, 0\right) \qquad (2.1)$$

for

$$t_{k-1} \le t < t_k, \qquad t_k = \sum_{i=1}^{k} \tau_i, \qquad k = 1, \cdots, N.$$

Here A is the node-edge connectivity matrix of dimension $|V| \times |E|$, i.e., the element a_{ij} of A is equal to 1 if the directed edge j originates at node i, to -1 if it terminates at node i, and to 0, otherwise. The quantity $q(t)$ is obtained by the indicated max since the activation of an intermediate link before any information arrives at its origin node does not effect any information transfer and leaves, therefore, the total information distribution unchanged. We refer to this phenomenon whenever it may occur as the "underflow" problem. We consider arbitrary topologies for the graph of the PRN. However, we classify the set of nodes of the PRN as follows (see Fig. 1). We let V_S denote the set of "originator" nodes, i.e., the nodes that, initially, possess information; we let V_D be the set of destination nodes, and we assume V_S and V_D to be disjoint. Finally, we denote by V' the set of remaining nodes. The interpretation of this classification is the following. First of all it permits us to consider more than one destination nodes without considering necessarily more commodities (a commodity is a "type" of information that is indexed by a distinct destination node). It represents a slight generalization over the case of a single destination node. In our case all nodes in V_D are equivalent in the sense that any information that reaches an arbitrary member of V_D is considered to have been "delivered" or "evacuated." Of course this model corresponds to the case where in reality we do have a single destination node which, however, is connected with dedicated links to every node in the set V_D. Secondly, the set V_S permits us to model the real sources of information (e.g., terminals or other devices) as separate node entities, each of which is separately attached to a network node that belongs to V'. Note that, as shown in Fig. 1, each of the nodes in either set V_D or V_S is connected to some node of the set V', which has a totally unrestricted topology.

It so happens that this classification does not diminish the generality of an arbitrary network graph (since our interpretation is legitimate) while at the same time it achieves a special structure for the graph that allows the use of crucially useful properties later on.

The equations in (2.1) that correspond to the destination nodes are redundant since they can be simply obtained from the rest; thus, in the following, we consider only those equations that correspond to nodes in V_S and V'. We can now state our problem as

$$\inf_{s \in S} L(s), \qquad (P)$$

where S is the set of all schedules such that $q(L(s)) = \mathbf{0}$ when $q(0) = q_0$.

Remark: Note that problem (P) is a dynamic optimization problem of high complexity. The objective function $L(\cdot)$ takes values in the set S that can be large and ill-structured. The main result of the correspondence is that problem (P) can be reduced to two static (finite-dimensional) optimization problems of reduced complexity.

For every schedule s define the *link activation vector* f_s as

$$f_s \triangleq \sum_{j=1}^{N} \tau_j I^{T_j}.$$

The ith element of this vector indicates the total activation time of link i. Now for each link activation vector f consider the set of

[1] A matching in graph G is a subset of its edges that contain no pair of adjacent edges.

[2] The max in (2.1) is taken component-wise.

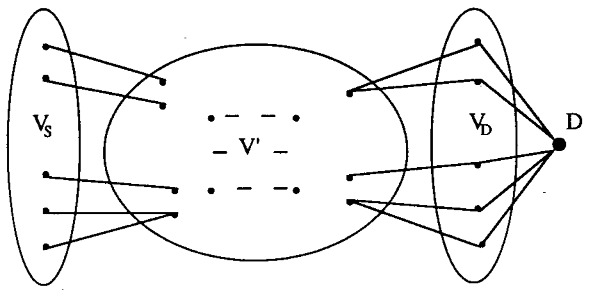

Fig. 1. Topological structure of the PRN.

schedules S_f defined as

$$S_f = \{s : f = f_s\}.$$

Notice that the schedules in S_f do not necessarily belong to S.

Furthermore, consider the set F of link activation vectors defined as

$$F = \{f : q_0 = Af\}.$$

We can state now our main result.

Theorem 1: The optimization problem (P) is decomposable into two optimization subproblems as

$$\inf_{s\in S} L(s) = \min_{f\in F}\left\{\min_{s\in S_f} L(s)\right\}.$$

Before proceeding to the proof of the theorem, we give some definitions and two lemmas.

A schedule s in S is defined to be *q_0-admissible* if it never activates links that cause the "underflow" phenomenon to occur when $q(0) = q_0$; that is, such a schedule ensures that $q(t) \ge 0$ at all times t, without taking the "max" as indicated in (2.1). Let S_{q_0} denote the set of *q_0-admissible* schedules. For such a schedule (2.1) becomes

$$q(0) = q_0,$$
$$q(t) = q(t_{k-1}) - (t - t_{k-1})AI^{T_k}, \quad \text{for } t_{k-1} \le t \le t_k, \tag{2.2}$$

where

$$t_0 = 0, \quad t_k = \sum_{j=1}^{k} \tau_j, \quad k = 1,\cdots,N \quad \text{and } q(t) \ge 0, \quad \forall t \ge 0.$$

Let us now consider a variation of problem (P) by restricting the schedule to be q_0-admissible, that is,

$$\inf_{s\in S_{q_0}} L(s).$$

For every schedule in S, we can find a q_0-admissible one with the same duration just by interrupting the operation of those links that cause underflow; thus the restriction of the optimization problem to admissible schedules does not increase the optimal value of the length. This is proven in the following lemma.

Lemma 1:

$$\inf_{s\in S} L(s) = \inf_{s\in S_{q_0}} L(s).$$

Proof: It is clear that $\inf_{s\in S} L(s) \le \inf_{s\in S_{q_0}} L(s)$ since $S_{q_0} \subset S$. Thus, it suffices to show that $\inf_{s\in S_{q_0}} L(s) \le \inf_{s\in S} L(s)$. For each schedule s, we will construct a q_0-admissible schedule s' with the same length that achieves the same final information distribution in the network as s. We construct the schedule s' as follows: if during transmission slot k of schedule s the information at node i becomes equal to zero at time t, we break the duration τ_k of transmission slot k into two parts. During the first part, i.e., during the interval $(\sum_{i=1}^{k-1} \tau_i, t)$ we activate the transmission set T_k, and during the second part, i.e., the remaining interval $(t, \sum_{i=1}^{k} \tau_i)$, we activate the transmission set T'_k that is the same as T_k except that the edge that was adjacent to node i and caused the underflow is now removed. By doing this adjustment to s for every instance of underflow, we eventually obtain a schedule s' that is by construction q_0-admissible. Furthermore, s' has obviously the same length as s and results in the same information distribution on the network nodes. □

Now, note that by summing (2.2) for $t = t_1,\cdots,t_N$, we clearly obtain

$$q_0 = Af_s, \tag{2.3}$$

provided s is q_0-admissible. Let now $S_{q_0}(f)$ be the set of q_0-admissible schedules that have a particular vector f as their link activation vector, i.e.,

$$S_{q_0}(f) = \{s : s \text{ is } q_0\text{-admissible and } f_s = f\}.$$

The following lemma is crucial for the proof of the theorem.

Lemma 2:

$$\inf_{s\in S_{q_0}(f)} L(s) = \inf_{s\in S_f} L(s).$$

Proof: We will show that

$$\forall s' \in S_f, \forall \epsilon > 0, \exists s \in S_{q_0}(f) \text{ such that } L(s) \le L(s') + \epsilon.$$

Let $s' = [(T'_i, \tau'_i), i = 1,\cdots,N]$ be a schedule such that $q_0 = Af'_s$. Assume for the moment that each node v in V' has nonzero initial amount of information δ_v. Let

$$d_v \triangleq \sum_{k=1}^{N} \tau'_k \sum_{e\in E(v)} I_e^{T'_k}, \tag{2.4}$$

where $E(v)$ is the set of links originating at v; the quantity d_v, therefore, represents the total amount of information that will depart from node v during the execution of schedule s'. Consider a schedule $\tilde{s}$ that is identical to s' except that the activation duration times $\tilde{\tau}_i$ are

$$\tilde{\tau}_i = \frac{\tau'_i}{K},$$

where K is a constant sufficiently large so that

$$\delta_v \ge \tilde{d}_v, \qquad \forall v \in V', \tag{2.5}$$

where $\tilde{d}_v$ is defined as in (2.4) but for the new schedule $\tilde{s}$. Relation (2.5) ensures that we can apply the schedule $\tilde{s}$ without risk of underflow. From the definition of $\tilde{s}$ it follows that

$$L(\tilde{s}) = \frac{L(s')}{K},$$

and that the amounts of information will change by $(1/K)q_s(0)$ in the nodes of V_S, and by zero in the nodes of V'. That is, with schedule $\tilde{s}$ the information in the source nodes will be reduced by $1/K$ of the amount by which it would be reduced if s' was applied and the length of the schedule $\tilde{s}$ will be K times less than that of s'. In addition, (2.5) ensures that underflow does not occur during the execution of S_δ. Also, since the information in the intermediate nodes (i.e., those of V') remains unchanged during the execution of the schedule s_δ, that schedule can be repeated. Consider now a schedule that consists of K repetitions of the schedule $\tilde{s}$; obviously this repetition schedule has length equal to $L(s')$ and transfers all of the information from the source nodes to the destinations; furthermore, it has link activation vector equal to f. What is left is the amount of information δ_v that we assumed was residing in each node of V'. Since the only assumption about that information was that it be greater than zero, we can take it to be arbitrarily small. Thus, we can consider that these amounts δ_v are transferred to the nodes of V' from the origin nodes V_S before we start the execution of $\tilde{s}$ and that we transfer them afterwards to the destinations to complete the total evacuation via an arbitrary trivial schedule that has length arbitrarily small (since δ_v can be as small as desired) and link activation vector f. Thus, the final schedule s is the schedule that consists of the concatenation of the schedule that transfers the initial amounts of information to the nodes of V' from the source nodes, the K repetitions of $\tilde{s}$ and the schedule that transfers to the destinations the information remaining in the nodes of V'. □

We can proceed now with the proof of the theorem.

Proof of Theorem 1: Since every q_0-admissible schedule satisfies (2.3), we have

$$S_{q_0} = \bigcup_{f \in F} S_{q_0}(f).$$

Hence, we have

$$\inf_{s \in S_{q_0}} L(S) = \min_{f \in F} \left\{ \inf_{s \in S_{q_0}(f)} L(s) \right\}. \tag{2.6}$$

From (2.6) and in view of Lemmas 1 and 2, we obtain

$$\inf_{s \in S} L(s) = \min_{f \in F} \left\{ \inf_{s \in S_f} L(s) \right\}. \tag{2.7}$$

The infimum on the right-hand side of (2.7) can be actually achieved by a schedule in S_f as we show in the following. Consider all possible transmission sets $T_1, \cdots, T_N$ of the network. For every schedule $s' \in S_f$, $s' = \{(\tau_i', T_i'),\ i = 1, \cdots, M'\}$, consider the schedule $s = \{(\tau_i, T_i),\ i = 1, \cdots, M\}$, where τ_i is the sum of those τ_j''s for which the corresponding T_j''s are the same set T_i. Clearly, we have

$$L(s) = L(s') \text{ and } f_s = f_{s'}.$$

Therefore, $s \in S_f$. Thus, it follows that the solution of the optimization problem (P') defined as

$$\min \sum_{i=1}^{N} \tau_i$$

subject to (P')

$$\sum_{i=1}^{M} \tau_i T_i = f, \qquad \tau_i \geq 0, \qquad i = 1, \cdots, M,$$

is equal to $\inf_{s \in S_f} L(s)$ and the τ_i's that achieve the minimum provide the optimal schedule. □

Remark: In order to obtain the optimal value in (P) we need to solve (P'). After we obtain the optimal value in (P') as a function of f, we optimize further by choosing $f \in F$. These two optimization problems have been studied in [1] and algorithms for their solution have been proposed.

III. Conclusion

The results in this correspondence can be useful in the process of topological design of a Packet Radio Network. There are still important problems associated with joint routing and scheduling that remain unaddressed. Specifically, the case of unequal link capacities, the case of multiple commodities that need to be routed, and, most importantly, the case of not evacuation but, rather, sustained network operation under random message generation remain unresolved and, largely, unaddressed.

References

[1] B. Hajek and G. Sasaki, "Link scheduling in polynomial time," *IEEE Trans. Inform. Theory*, vol. 34, pp. 910-917, Sept. 1989.

[2] B. Hajek, "Link schedules, flows, and the multicommodity index of graph," in *Proc. 1984 Conf. Inform. Sci. Syst.*, Mar. 1984, pp. 498-502.

[3] D. J. Baker, J. Wieselthier, and A. Ephremides, "A distributed algorithm for scheduling the activation of links in a self-organizing mobile radio network," in *IEEE Int. Conf. Commun. Conf. Rec.*, Philadelphia, PA, June 1982, pp. 2F. 6.1-2F. 6.5.

[4] L. Tassiulas, "Scheduling problems in packet radio networks," M.S. thesis, Univ. of Maryland, College Park, 1989.

[5] J. Cidon and M. Sidi, "Distributed assignment algorithms for multihop packet radio networks," *IEEE Trans. Comput.*, vol. 38, Oct. 1989.

[6] M. J. Post, A. S. Kershenbaum, and P. E. Sarachik, "Scheduling multi-hop CDMA networks in the presence of secondary conflicts," Preprint.

[7] L. Kleinrock, *Queueing Systems, Vol. II: Computer Applications.*

The Zak Transform and Some Counterexamples in Time-Frequency Analysis

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Abstract—**It is shown how the Zak transform can be used to find nontrivial examples of functions $f, g \in L^2(\mathbb{R})$ with $f \cdot g \equiv 0 \equiv F \cdot G$, where F, G are the Fourier transforms of f, g, respectively. This is then used to exhibit a nontrivial pair of functions $h, k \in L^2(\mathbb{R})$, $h \neq k$, such that $|h| = |k|$, $|H| = |K|$. A similar construction is used to find an abundance of nontrivial pairs of functions $h, k \in L^2(\mathbb{R})$, $h \neq k$, with $|A_h| = |A_k|$ or with $|W_h| = |W_k|$, where A_h, A_k and W_h, W_k are**

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the ambiguity functions and Wigner distributions of h, k, respectively. One of the examples of a pair of $h, k \in L^2(\mathbb{R})$, $h \neq k$, with $|A_h| = |A_k|$ is F. A. Grünbaum's example given previously. We find, in addition, nontrivial examples of functions g and signals $f_1 \neq f_2$ such that f_1 and f_2 have the same spectrogram when using g as window.

Index Terms—**Zak transform, ambiguity function, spectrogram.**

I. Introduction

In [1], F. A. Grünbaum presents a nontrivial example of two functions $h, k \in L^2(\mathbb{R})$ such that $|A_h| = |A_k|$. By nontrivial we mean that h and k cannot be obtained from one another by a time-frequency translate or by multiplication by a $c \in \mathbb{C}$, $|c| = 1$. Here, A refers to the ambiguity function: when $f, g \in L^2(\mathbb{R})$, the ambiguity function $A_{f,g}$ of f and g is defined by

$$A_{f,g}(\theta,\tau) = \int_{-\infty}^{\infty} e^{-2\pi i\theta s} f\left(s+\frac{1}{2}\tau\right) g^*\left(s-\frac{1}{2}\tau\right) ds, \qquad \theta,\tau\in\mathbb{R}. \quad (1.1)$$

When $f = g$, we write A_f instead of $A_{f,f}$. The purpose of this note is to show that Grünbaum's example is a particular case of a whole class of such examples that can be constructed by using the Zak transform. A second purpose is to present similar examples, by similar constructions, of Fourier pairs, spectrograms and Wigner distributions.

The Zak transform of an $f \in L^2(\mathbb{R})$ is defined by

$$(Zf)(\tau,\Omega) = \sum_{k=-\infty}^{\infty} e^{-2\pi ik\Omega} f(\tau+k), \qquad \tau,\Omega\in\mathbb{R}. \quad (1.2)$$

We recall here the properties of the Zak transform needed for the present purposes. We have the following, cf. [2].

1) Z is a Hilbert space isometry of $L^2(\mathbb{R})$ onto $L^2([-\frac{1}{2},\frac{1}{2}]^2)$. More precisely, when $Z(\tau,\Omega)$ is a function satisfying

$$Z(\tau+1,\Omega) = e^{2\pi i\Omega} Z(\tau,\Omega),$$
$$Z(\tau,\Omega+1) = Z(\tau,\Omega), \qquad \tau,\Omega\in\mathbb{R}, \quad (1.3)$$

and $Z \in L^2([-\frac{1}{2},\frac{1}{2}]^2)$, there is exactly one $f \in L^2(\mathbb{R})$ such that $Z = Zf$. Conversely, $Zf \in L^2([-\frac{1}{2},\frac{1}{2}]^2)$ and Zf satisfies the (quasi) periodicity relations (1.3) when $f \in L^2(\mathbb{R})$. And

$$(Zf, Zg) = (f, g), \qquad f, g \in L^2(\mathbb{R}), \quad (1.4)$$

where the left-hand side inner product is that in $L^2([-\frac{1}{2},\frac{1}{2}]^2)$ and the right-hand side inner product is that in $L^2(\mathbb{R})$.

2) For $f \in {}^2(\mathbb{R})$ we have the formulas

$$f(\tau) = \int_0^1 (Zf)(\tau,\Omega)\, d\Omega, \qquad F(\omega) = \int_0^1 e^{-2\pi i\tau\omega}(Zf)(\tau,\omega)\, d\tau, \qquad t,\omega\in\mathbb{R}, \quad (1.5)$$

where F denotes the Fourier transform of f,

$$F(\omega) = \int_{-\infty}^{\infty} e^{-2\pi i\omega t} f(t)\, dt, \qquad \omega\in\mathbb{R}. \quad (1.6)$$

3) For $f, g \in L^2(\mathbb{R})$ we have the formula

$$(Zf)(\tau,\Omega)(Zg)^*(\tau,\Omega) = \sum_{n,m} (f, R_{-m}T_{-n}g) e^{-2\pi in\Omega+2\pi im\tau}, \quad (1.7)$$

where for $a, b \in \mathbb{R}$ the operators T_a, R_b are time, frequency shifts defined by

$$(T_af)(t) = f(t+a), \qquad (R_bf)(t) = e^{-2\pi ibt} f(t), \qquad t\in\mathbb{R}. \quad (1.8)$$

4) We have for $f \in L^2(\mathbb{R})$, $a, b \in \mathbb{R}$,

$$(ZT_af)(\tau,\Omega) = (Zf)(\tau+a,\Omega), \qquad (ZR_bf)(\tau,\Omega) = e^{-2\pi ib\tau}(Zf)(\tau,\Omega+b). \quad (1.9)$$

Formula (1.7) provides an important link between the Zak transform and the ambiguity function since

$$A_{f,g}(\theta,\tau) = e^{\pi i\theta\tau}(f, R_{-\theta}T_{-\tau}g), \qquad \theta,\tau\in\mathbb{R}. \quad (1.10)$$

II. The Examples

Example 1: $f, g \in L^2(\mathbb{R})$ such that $f \cdot g \equiv 0 \equiv F \cdot G$.

Let U and V be two subsets of $[-\frac{1}{2},\frac{1}{2}]^2$ such that for any $\tau,\Omega \in [-\frac{1}{2},\frac{1}{2}]$

$$\mu(U_\tau)\mu(V_\tau) = \mu(U^\Omega)\mu(V^\Omega) = 0. \quad (2.1)$$

Here,

$$U_\tau = \{\Omega | (\tau,\Omega)\in U\}, \qquad U^\Omega = \{\tau | (\tau,\Omega)\in U\}, \quad \text{etc.}, \quad (2.2)$$

and μ is Lebesgue measure on $[-\frac{1}{2},\frac{1}{2}]$. Let $\varphi, \psi \in L^2([-\frac{1}{2},\frac{1}{2}]^2)$ have their supports in U, V, respectively, and extend φ, ψ quasi-periodically according to (1.3) to all of $\mathbb{R}^2$. Then $\varphi = Zf$, $\psi = Zg$ for some $f, g \in L^2(\mathbb{R})$, and, as readily follows from 2), we have $f \cdot g \equiv 0 \equiv F \cdot G$.

Note: In terms of ambiguity functions we have here an example of an f, g such that $A_{f,g}(0,\tau) = A_{f,g}(\theta,0)$ for all $\theta,\tau \in \mathbb{R}$. That $A_{f,g}$ cannot vanish identically follows from

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |A_{f,g}(\theta,\tau)|^2\, d\theta\, d\tau = \|f\|^2\|g\|^2. \quad (2.3)$$

Example 2: $h, k \in L^2(\mathbb{R})$, $h \neq k$, such that $|h| = |k|$, $|H| = |K|$.

It is easy to find such h, k as follows. Let $h \in L^2(\mathbb{R})$ be such that $|h(t)| = |h^\star(-t)|$, and set $k(t) = h^\star(-t)$. Then $K(\omega) = H^\star(\omega)$, so that $|K| = |H|$. A less trivial example is obtained by setting $h = f + g$, $k = f - g$, with f, g as in Example 1, so that $|h| = |f| + |g| = |k|$, $|H| = |F| + |G| = |K|$.

Example 3: For the next set of examples we need a lemma on the supporting sets of ambiguity functions.

Lemma 1: Denote for $f \in L^2(\mathbb{R})$ by S_f the supporting set of Zf (by (1.3) this set is periodic in both variables). Furthermore, denote for $f \in L^2(\mathbb{R})$ and $\tau_0, \Omega_0 \in [-\frac{1}{2}, \frac{1}{2}]$ by $S_f(\tau_0, \Omega_0)$ the set

$$S_f(\tau_0, \Omega_0) = \{(\tau_0 + \tau, \Omega_0 + \Omega) | (\tau, \Omega) \in S_f\}. \quad (2.4)$$

Finally, denote for $f, g \in L^2(\mathbb{R})$ by $\sum_{f,g}$ the set

$$\sum_{f,g} = \left\{(\tau_0, \Omega_0) \in [-\tfrac{1}{2}, \tfrac{1}{2}]^2 \middle| \mu_2(S_f \cap S_g(\tau_0, \Omega_0)) \neq 0\right\}. \quad (2.5)$$

Here, μ_2 is Lebesgue measure in $\mathbb{R}^2$. Then $A_{f,g}$ is supported by the set $V_{f,g}$ given by

$$V_{f,g} = \left\{(n + \tau_0, m + \Omega_0) \middle| n, m \in \mathbb{Z}, (\tau_0, \Omega_0) \in \sum_{f,g}\right\}. \quad (2.6)$$

Proof: We combine (1.7), (1.9), and (1.10) to obtain

$$\begin{aligned} e^{-2\pi i \Omega_0 \tau}(Zf)(\tau, \Omega)(Zg)^\star(\tau - \tau_0, \Omega - \Omega_0) &= \sum_{n,m} e^{-\pi i (m + \Omega_0)(n + \tau_0)} \\ &\quad A_{f,g}(m + \Omega_0, n + \tau_0) e^{2\pi i (m\tau - n\Omega)}, \end{aligned} \quad (2.7)$$

for $\tau, \tau_0, \Omega, \Omega_0 \in \mathbb{R}$. Now we have for $\tau_0, \Omega_0 \in [-\frac{1}{2}, \frac{1}{2}]$ that

$$A_{f,g}(m + \Omega_0, n + \tau_0) = 0, \qquad \text{all } n, m \in \mathbb{Z}, \quad (2.8)$$

if and only if

$$\mu_2(S_f \cap S_g(\tau_0, \Omega_0)) = 0. \quad (2.9)$$

Since any point $(\theta, \tau) \in \mathbb{R}^2$ can be written as $(m + \Omega_0, n + \tau_0)$ for some $n, m \in \mathbb{Z}$, $\tau_0, \Omega_0 \in [-\frac{1}{2}, \frac{1}{2}]$, the lemma follows. □

To give some insight how the lemma can be used to construct counterexamples, we present Figs. 1–4. Observe that $\sum_{f,g} = -\sum_{g,f}$, so that $\sum_{f,f}$ is symmetric about the origin.

With the aid of Lemma 1, one can construct functions f whose ambiguity function A_f has, in the terminology of Price and Hofstetter [3], volume-clearance around the origin arbitrarily close to 4. That is, for any $\delta > 0$, $\epsilon > 0$, there is an $f \in L^2(\mathbb{R})$ and a convex set C with $\mu_2(C) \geqq 4 - \delta$ such that $A_f(\theta, \tau) = 0$ for $(\theta, \tau) \in C$, $\theta^2 + \tau^2 \geqq \epsilon^2$. One can take for f any function whose Zak trans is concentrated in a small disk around the origin. The volume-clearance result in [3] says that $\mu_2(C)$ cannot exceed 4. As a limiting case, when $\epsilon \downarrow 0$, $\delta \downarrow 0$, one can take $f = \sum_n \delta_n$ so that $A_f(\theta, \tau) = \sum_{n,m} \delta_n(\theta)\delta_m(\tau)$ (here, δ_n is the delta function at n).

Example 4: $h, k \in L^2(\mathbb{R})$, $n \neq k$, such that $|A_h| = |A_k|$.

It is easy to see that we have $|A_h| = |A_k|$ when $h \in L^2(\mathbb{R})$ and $k = cR_bT_ah$ for some $a, b \in \mathbb{R}$, $c \in \mathbb{C}$, $|c| = 1$. Less trivial examples can be constructed as follows. Take $f = f_1$, $g = g$ as in Fig. 3. Now we have

$$\sum_{f,g} \cap \sum_{f,f} = \sum_{g,f} \cap \sum_{f,f} = \sum_{f,g} \cap \sum_{g,g} = \sum_{g,f} \cap \sum_{g,g} = \varnothing. \quad (2.10)$$

Hence, $A_{f,g} \cdot A_{f,f} = 0$, etc. When we set $h = f + g$, $k = f - g$ and observe that

$$A_{f \pm g, f \pm g} = A_{f,f} + A_{g,g} \pm (A_{f,g} + A_{g,f}), \quad (2.11)$$

we readily see that $|A_h| = |A_k|$. An example of this situation of

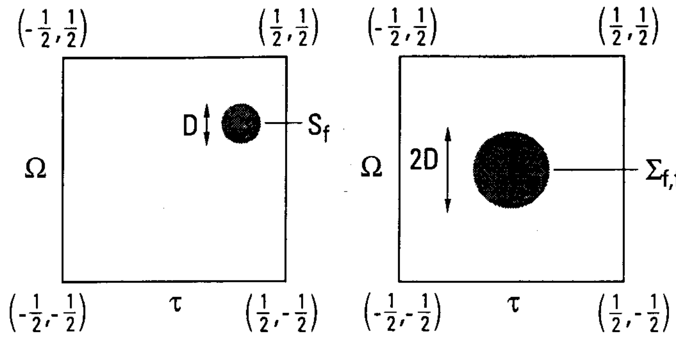

Fig. 1. S_f and $\sum_{f,f}$ for a monocomponent f in the Zak domain.

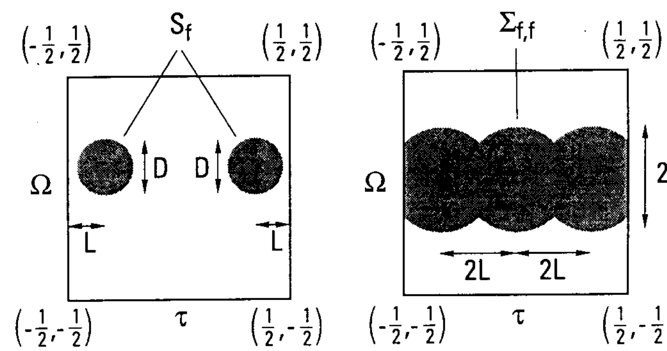

Fig. 2. S_f and $\sum_{f,f}$ for a multicomponent f in the Zak domain; the two components in $\sum_{f,f}$ lying symmetrically about the origin are due to periodicity of the set S_f.

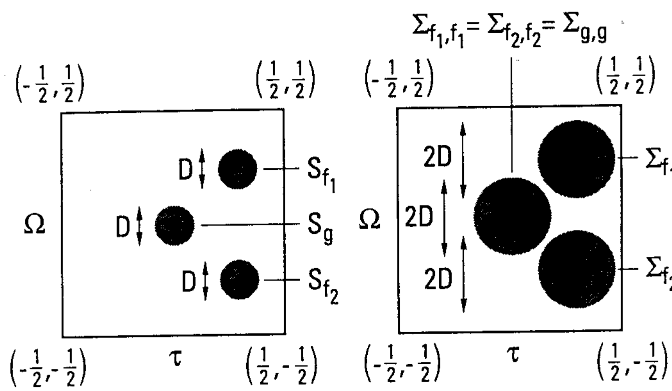

Fig. 3. S_g, S_{f_1} and S_{f_2} for three monocomponent functions in the Zak domain and the sets $\sum_{f_1,g}$, $\sum_{f_1,f_1}$, $\sum_{f_2,g}$, $\sum_{f_2,g}$, $\sum_{g,g}$.

the Grünbaum type is given in Fig. 4. Grünbaum considers functions f and g with support in $|t| \leqq 1$ and $4 \leqq |t| \leqq 5$, respectively. By appropriate translation and scaling, it can be achieved that f and g have their supports in intervals $(\epsilon, 2\epsilon)$ and $(\frac{1}{2} - 2\delta, \frac{1}{2} - \delta)$. It follows from the definition of the Zak transform that S_f and S_g are as in Fig. 4. Again, we have a situation in which (2.10) holds.

Example 5: $h, k \in L^2(\mathbb{R})$, $h \neq k$, such that $|W_h| = |W_k|$. When $f, g \in L^2(\mathbb{R})$ we define the Wigner distribution of f and g by

$$W_{f,g}(t, \omega) = \int_{-\infty}^{\infty} e^{-2\pi i s \omega} f(t + \tfrac{1}{2}s) g^\star(t - \tfrac{1}{2}s)\, ds, \qquad t, \omega \in \mathbb{R}. \quad (2.12)$$

Unlike the ambiguity function, $W_{f,f}$ is always real. We have

$$W_{f,g}(t, \omega) = 2A_{f,g_-}(2\omega, 2t), \qquad t, \omega \in \mathbb{R}, \quad (2.13)$$

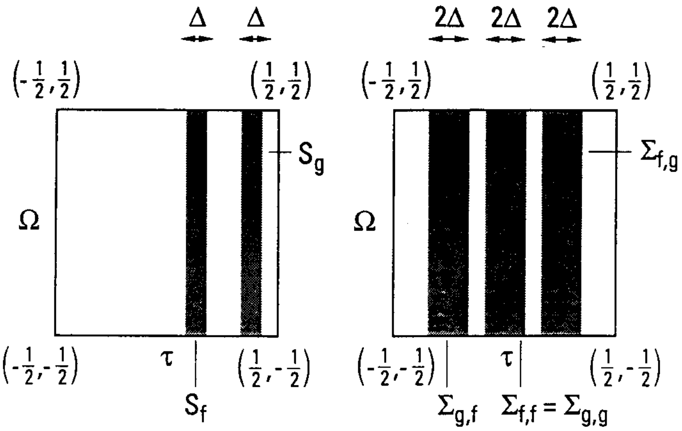

Fig. 4. S_f, S_g, $\Sigma_{f,f}$, $\times_{f,g}$, $\Sigma_{g,f}$ for two functions concentrated in a strip in the Zak domain.

where $g_-(t) = g(-t)$. Noting that $(Zg_-)(\tau, \Omega) = (Zg)(-\tau, \Omega)$, we can easily modify the argument in Example 4 to construct examples of $h, k \in L^2(\mathbb{R})$ such that $|W_h| = |W_k|$. In particular, F. A. Grünbaum's function provides such an example.

Example 6: h, k, g such that $h \neq k$ have the same spectrogram using g as window.

When g is a window function, the spectrogram of $f \in L^2(\mathbb{R})$ is defined as

$$\left| \int_{-\infty}^{\infty} e^{-2\pi i s \omega} g(t-s) f(s)\, ds \right|^2 = \left| A_{f,\tilde{g}}(\omega, t) \right|^2, \qquad t, \omega \in \mathbb{R}, \tag{2.14}$$

where $\tilde{g} = g_-^{\star}$. In Fig. 3, we have f_1, f_2 such that

$$\sum_{f_1, g} \cap \sum_{f_2, g} = \varnothing, \tag{2.15}$$

hence $A_{f_1,g} \cdot A_{f_2,g} = 0$. When we take $h = f_1 + f_2$, $k = f_1 - f_2$, and replace g by $\tilde{g}$, we see that h and k have the same spectrograms.

References

[1] F. A. Grünbaum, "A remark on the radar ambiguity function," *IEEE Trans. Inform. Theory*, vol. 30, pp. 126–127, Jan. 1984.

[2] A. J. E. M. Janssen, "The Zak transform: a signal transform for sampled time-continuous signals, *Philips J. Res.*, vol. 43, 1988.

[3] R. Price and E. M. Hofstetter, "Bounds on the volume and height distributions of the ambiguity function," *IEEE Trans. Inform. Theory*, vol. 11, pp. 207–214, Apr. 1965.

On Convergence of Lloyd's Method I

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Abstract—**Although Lloyd's method I for optimal quantization was proposed more than thirty years ago and has been frequently referred to in the literature, its convergence has so far not been shown. This correspondence proves that Lloyd's method I converges for a large class of error measures, if the density function is continuous, positive, and defined on a finite interval. The proof is done by modeling the behavior of a continuous optimization algorithm by a finite state machine.**

Index Terms—**Convergence of sequences, optimal quantization, fixed point iteration, finite state machine.**

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I. Introduction

The well-known Lloyd's method I [7] for optimal quantization is a fixed point algorithm to compute a locally optimal quantizer. The method was originally derived for the mean-square error measure, but is applicable for a wide range of error measures, as we will see later. After its invention in 1957, Lloyd's method I was extended by Netravali and Saigal [9] for optimal quantization under entropy constraints. Then the fixed point iteration scheme of Lloyd's method I was generalized from scalar quantization to vector quantization, resulting in the popular Linde–Buzo–Gray (LBG) algorithm [6].

Despite its long history in use no one, to the best of author's knowledge, has proven the convergence of Lloyd's method I before. Interestingly, the convergence of the LBG algorithm, the vector version of the scalar Lloyd's method I, was shown by Abaya and Wise [1], by Selim and Ismail [11] when they proved the convergence of the K-means algorithm, and later by Sabin and Gray [10] in a more general setting. It should be noted though that the LBG algorithm is by nature one of discrete optimization. Being iteratively applied to an initial code book the LBG algorithm generates a sequence of ever-improved code books. All these code books contain a finite number of words (points in a vector space). A code book which may be perceived as a vector quantizer is a partition of a finite point set, hence the both sets of input and output for the LBG algorithm are finite. The original Lloyd's method I is, on the contrary, a continuous optimization algorithm, trying to partition an infinite number of points obeying a continuous density function $p(x)$ into K sets. Due to this significant difference, the proofs of convergence cited previously for the LBG algorithm cannot be extended to the original Lloyd's method I.

The convergence of Lloyd's method I was previously studied by a number of researchers [2], [5], [12] in the context of uniqueness of a locally optimal quantizer. It was shown that Lloyd's method I converges to the globally optimal quantizer if the density function is continuous and log-concave. and if the error weighting function is convex and symmetric. In this correspondence, we will prove that Lloyd's method I converges for all continuous, positive densities defined on a finite interval under the class of convex and symmetric error measures. This more general result is obtained by a finite state machine that models the behavior of Lloyd's method I and by using a monotonicity property of the method.

II. Formulation and Preparation

In order to facilitate the key proof of the correspondence, we need to formulate the problem of optimal quantization, and list some published results about the problem. It is assumed that the signal amplitude density function $p(x)$ is continuous, positive, and defined on a finite interval which is normalized to [0, 1], that is, $(\forall x, x \in [0, 1]) p(x) > 0$, $(\forall x, x \notin [0, 1]) p(x) = 0$ and $\int_0^1 p(x)\, dx = 1$. Concisely, a K-level quantizer for $p(x)$ may be characterized by two vectors $q \in \Re^{K-1}$, $q_{j-1} < q_j$, $1 \le j \le K$, and $r \in \Re^K$. The vector q partitions the range [0, 1] into K intervals: $[0, q_1]$, $(q_{j-1}, q_j]$ for $1 < j < K-1$, and $(q_{K-1}, 1]$. Here the conventions $q_0 \equiv 0$ and $q_K \equiv 1$ come naturally and will be used in the sequel.