DISTRIBUTION OF PATH DURATIONS IN MOBILE AD-HOC NETWORKS – PALM'S THEOREM AT WORK

Yijie Han, Richard J. La, and Armand M. Makowski
Department of ECE and ISR
University of Maryland, College Park
hyijie@eng.umd.edu, {hyongla,armand}@isr.umd.edu

ABSTRACT

We first study the distribution of path duration in multi-hop wireless networks. We show that as the number of hops along a path increases, the path duration distribution can be accurately approximated by an exponential distribution under a set of mild conditions, even when the link duration distributions are not identical. Then, we develop an approximate model for the Random Waypoint (RW) mobility model for computing the distribution of link duration, and demonstrate that the path duration distribution converges to an exponential distribution with increasing number of hops.

1. Introduction

Routing protocols for multi-hop wireless ad-hoc networks are classified as being either table-driven or on-demand. Table-driven routing protocols attempt to maintain a path between any two nodes at all times, whereas on-demand routing protocols establish a path between two nodes only upon request. Due to node mobility, links along provided paths may become unavailable in an unpredictable manner, a situation which would trigger path recovery. Thus, as the performance of various on-demand routing protocols and their overheads are likely to be shaped by the distribution of link and path durations, there is a need to better understand their characteristics. Indeed, accurate modeling of these link and path durations can help better evaluate the performance of current and new on-demand routing protocols without having to run time-consuming detailed simulations.

These distributions are expected to depend on the mobility models used in the simulations as well as on the range of node speeds. Along these lines, Sadagopan *et al.* [8] have recently presented a numerical study of the distribution of multi-hop path durations under various mobility models. Their study shows that the distribution of path duration can be accurately approximated by an exponential distribution when the number of hops is larger than 3 or 4 for *all* mobility models considered. However, no explanation was offered for the emergence of the exponential distribution. In this paper, we develop an approximate framework for handling this issue and use it to *prove* that under a set of mild conditions, when the number of hops becomes large, the distribution of path duration can indeed be accurately approximated by an exponential distribution. These results are simply another incarnation of Palm's Theorem [5, Thm. 5-14, p. 157], the one-dimensional precursor of the celebrated Palm-Khintchine Theorem [5, Thm. 5-15, p. 160] to the effect that the superposition of a large number of independent equilibrium renewal processes, each with a small intensity, behaves asymptotically like a Poisson process. We validate our results through an approximate model we develop for Random Waypoint (RW) mobility model in order to compute the distribution of link and path durations with varying number of hops.

The paper is organized as follows. We describe the model for studying a path duration in Section 2.. Section 3. presents the convergence results of the path duration distribution. A numerical example is provided in Section 4. using a RW mobility model. Section 5. provides a justification for our assumption in Section 3..

A word on the notation and convention used throughout: We find it convenient to define all the random variables (rvs) of interest on some common probability space $(\Omega, \mathcal{F}, \mathbf{P})$. Two $\mathbb{R}$–valued rvs X and Y are said to be *equal in law* if they have the same distribution, a fact we denote by $X =_{st} Y$. For any $\alpha > 0$, we denote by $\mathcal{E}_\alpha$ any rv that is exponentially distributed with parameter α, *i.e.*,

$$\mathbf{P}\left[\mathcal{E}_\alpha \le x\right] = \begin{cases} 1 - e^{-\alpha x} & \text{if } x > 0 \\ 0 & \text{if } x \le 0. \end{cases} \qquad (1)$$

All rvs will be $\mathbb{R}_+$-valued rvs. If H is a probability distribution on $\mathbb{R}_+$, let $m(H)$ denote its first moment which is always assumed to be finite. Convergence in distribution as $n \uparrow \infty$ is denoted by $\Longrightarrow_n$.

2. A basic framework

Consider a mobile ad-hoc network where a set of nodes creates and maintains a network connectivity. The routing algorithm is assumed to be an on-demand algorithm, *i.e.,* a path between a source (node) and a destination (node) is set up only when a request is made. A detailed discussion of available on-demand routing protocols is outside the scope of this paper, and we refer the interested reader to the monographs [7, 9] for additional information regarding these routing protocols.

Let $V = \{1, \ldots, \text{I}\}$ denote the set of mobile communicating nodes. Each node moves across a domain D of $\mathbb{R}^2$ or $\mathbb{R}^3$ according to some mobility model. Since there is no fixed infrastructure and nodes are mobile, links between nodes are set up and torn down dynamically. We assume that two nodes i and j become neighbors and establish a link between them when the distance becomes smaller than some constant $r_{min} > 0$, and the link is torn down when this distance becomes larger than r_{min}. The link is said to be up in the former case, and down in the latter.

The establishment of a path from a source node to a destination node requires the simultaneous availability of links that are up, one originating at the source node and another ending at the destination node, together providing connectivity between the source and the destination. The *path duration* is then defined as the length of time that elapses from the moment the path is established until that time when one of the links along the path goes down, as a result of mobility or interferences. For simplicity of analysis we assume path setup delays to be negligible.

We model this situation as follows: To account for mobility, for i and j distinct in V, we introduce a $\{0,1\}$-valued *reachability* process $\{\xi_{ij}(t),\ t \geq 0\}$ with the interpretation that $\xi_{ij}(t) = 1$ (resp. $\xi_{ij}(t) = 0$) if the "link" (i, j) is up (resp. down) at time $t \geq 0$. The process $\{\xi_{ij}(t),\ t \geq 0\}$ is simply an alternating on-off process, with successive up and down time durations given by the rvs $\{U_{ij}(k), k = 1, 2, \ldots\}$ and $\{D_{ij}(k), k = 1, 2, \ldots\}$, respectively. Since two nodes are neighbors of each other when their distance is smaller than r_{min}, the links are symmetric, *i.e.*, $\xi_{ij}(t) = \xi_{ji}(t)$.

Next we endow V with a time-varying graph structure by introducing a time-varying set of directed edges through the relation

$$E(t) := \{(i, j) \in V \times V : \ \xi_{ij}(t) = 1\}, \quad t \geq 0 \quad (2)$$

where by convention we have set $\xi_{ii}(t) = 0$ for each $i = 1, \ldots,$ I and all $t \geq 0$. Thus, a path can be established (in principle) between nodes i and j at time $t \geq 0$, if node j is reachable from node i in the *undirected* graph derived from the directed graph $(V, E(t))$. Let $L_{ij}(t)$ denote a set of links providing this reachability, *i.e.*, a path from node i to node j. This set of links is empty when the nodes i and j are not reachable from each other at time t. When non-empty, this set $L_{ij}(t)$ is not necessarily unique since multiple paths may exist between the pair of nodes i and j, in which case the routing protocol in use selects one of the possible paths.

Consider the example in Figure 1. Assume that a path is requested from node s to node d at some time $t \geq 0$. A dotted line between two nodes i and j indicates that the link between them is up, *i.e.*, $\xi_{ij}(t) = 1$. Since there exists more than one path from s to d, *e.g.*, $\{(s, n2), (n2, d)\}$ and $\{(s, n3), (n3, d)\}$, the underlying routing algorithm selects one of them. As there is no dotted line between s and d, *i.e.*, $\xi_{sd}(t) = 0$, no one-hop path can be established at the time of path request.

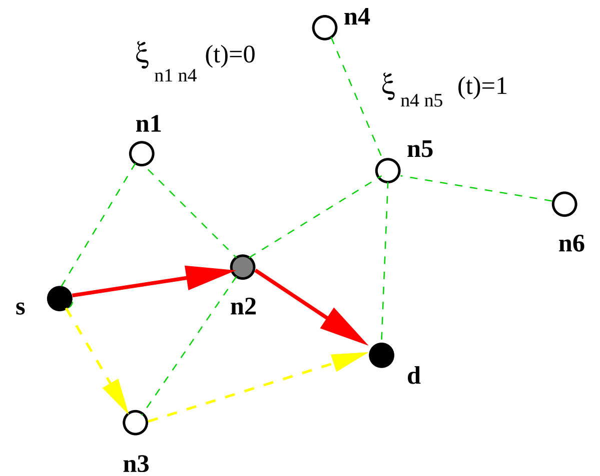


Figure 1: An example of a path and reachability.

For each link ℓ in $L_{ij}(t)$, let $T_\ell(t)$ denote the time-to-live or excess life after time t, *i.e.*, $T_\ell(t)$ is the time that elapses from time t onward until the first moment link ℓ is down. The time-to-live or duration $Z_{ij}(t)$ of the established path from node i to node j using the links in $L_{ij}(t)$ is defined as the amount of time that elapses from time t until one of the links in $L_{ij}(t)$ goes down. This quantity is simply given by

$$Z_{ij}(t) := \min\left(T_\ell(t) : \ell \in L_{ij}(t)\right), \quad t \geq 0. \quad (3)$$

In the example in Figure 1, if the routing algorithm were to select the path $L_{sd}(t) = \{(s, n2), (n2, d)\}$, then its duration would be given by $\min\left(T_{(s,n2)}(t),\ T_{(n2,d)}(t)\right)$ as depicted in Figure 2. Since the excess life of link $(n2, d)$ is smaller than that of link $(s, n2)$, the path duration is given by $T_{(n2,d)}(t)$.

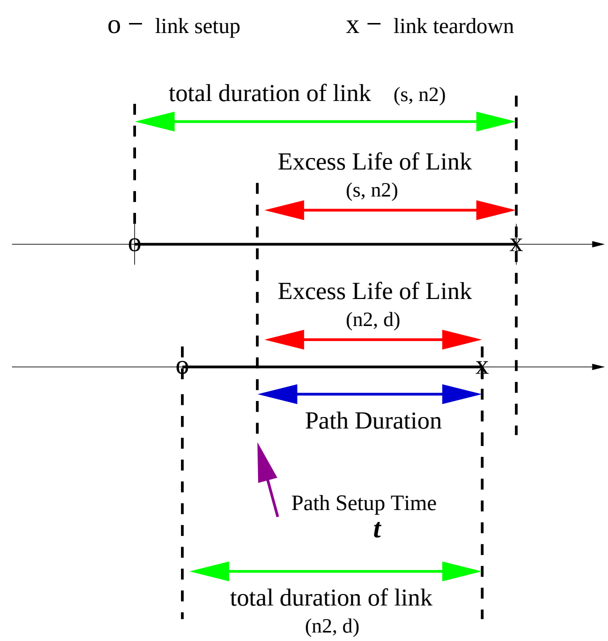


Figure 2: Link excess life and path duration.

As discussed in Section 1, the overall performance of an on-demand routing protocol is determined by the overhead it incurs, which in turn depends on the distribution of path durations. Hence, there is a great deal of interest in understanding the distributional properties of the rvs defined through (3). In order to make progress, we shall make several simplifying assumptions:

1. First we assume that the reachability processes $\{\xi_{ij}(t),\ t \geq 0\}$ $(i \neq j \in V)$, are *mutually independent*. Although this assumption is not true in general, numerical studies show that the dependency between two neighboring links is weak and that of two links separated by one or more links in-between is negligible. We refer the reader to Section 5 for the computation of correlation coefficient of excess lives of two neighboring links and to reference [4] for links separated by one or more links in-between.

2. Next, as the system is expected to run for a long time, we can expect steady state to be reached. We model this by taking each reachability process to be *stationary*, say with the rvs $\{(U_{ij}(k), D_{ij}(k)), k = 2, 3, , \ldots\}$ forming a strictly stationary sequence with generic marginals (U_{ij}, D_{ij}). We denote by G_{ij} the cumulative distribution function (CDF) of U_{ij}.

   Well-known results for renewal processes and independent on-off processes in equilibrium [5, Section 5-6] can be generalized as follows: With $\ell = (i, j)$ we have

$$\mathbf{P}\left[T_\ell(0) \leq x | \xi_{ij}(0) = 1\right] = F_\ell(x) \qquad (4)$$

   where F_ℓ is the CDF given by

$$F_\ell(x) = \begin{cases} \frac{1}{m(G_\ell)} \int_0^x (1 - G_\ell(y)) dy & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases} . \qquad (5)$$

   In other words, F_ℓ is simply the distribution of the forward recurrence time associated with D_ℓ. One can immediately see from (5) the duration of an one-hop path has a non-increasing probability density function (PDF). Throughout, let X_ℓ denote a rv distributed according to F_ℓ. The relation (4) simply states, with a little abuse of notation, that

$$[T_\ell(0) \leq x | \xi_{ij}(0) = 1] =_{st} X_\ell.$$

The rest of this section and the next section are devoted to exploring the distributional properties of the rvs defined through (3). In view of (4) this amounts to considering the rvs $\{Z_{ij}, i \neq j \in V\}$ defined by

$$Z_{ij} := \min\left(X_\ell :\ \ell \in L_{ij}(0)\right), \qquad (6)$$

where the rvs $\{X_\ell,\ \ell \in L_{ij}(0)\}$ are mutually independent given $L_{ij}(0)$.

3. Applying Palm's Theorem

Typically, the set $L_{ij}(0)$ will itself be random, depending on the relative locations of nodes i and j in the network and the underlying routing protocol. For example, if nodes are uniformly distributed on the surface of a sphere or on a disk, and the nodes i and j are selected randomly, then $\mathbf{E}\left[|L_{ij}(0)|\right] \propto \sqrt{\mathrm{I}}$, where I is the number of nodes in the network [3]. As a result, evaluating the exact distribution of (6) is difficult. Instead, we focus on the situation where the number of nodes increases while the area simultaneously increases, and study the distributional properties of the rv Z_{ij} as the number of links in $L_{ij}(0)$ increases.

Consider the following scenario. For each fixed $n = 1, 2, \ldots$, let $V^{(n)}$ and $D^{(n)}$ be the set of mobile nodes and the domain across which the nodes move. We assume that $|V^{(n)}|$ and area($D^{(n)}$) increase together with n.[1] For each $n = 1, 2, \ldots$, we select a pair of nodes i and j such that

$$|L_{ij}^{(n)}|/n \to \kappa \ \text{ as } \ n \to \infty \ \text{ for some positive constant } \kappa.$$

Under this assumption we are now faced with the problem of exploring the distributional properties of the rvs defined through

$$Z_{ij}^{(n)} := \min\left(X_\ell^{(n)} :\ \ell = 1, \ldots, |L_{ij}^{(n)}|\right) \qquad (7)$$

where the rvs $X_\ell^{(n)}, \ell = 1, \ldots, |L_{ij}^{(n)}|$, are mutually independent rvs distributed according to the distribution of the forward recurrence time associated with $D_\ell^{(n)}$. We are now ready to discuss the asymptotic behavior of (7) as $|L_{ij}^{(n)}|$ becomes large, and the emergence of the exponential distribution in the limit.

As we focus on a particular pair of source and destination nodes, we drop the subscript ij in the notation introduced earlier. Thus, let $\{X_\ell^{(n)},\ \ell = 1, \ldots, |L^{(n)}|;\ n = 1, 2, \ldots\}$ be an array of independent $\mathbb{R}_+$-valued rvs. For each $\ell = 1, \ldots, |L^{(n)}|$, the rv $X_\ell^{(n)}$ is distributed according to the distribution $F_\ell^{(n)}$ associated by (5) with some link duration distribution $G_\ell^{(n)}$. The distributions $\{G_\ell^{(n)},\ \ell = 1, \ldots, |L^{(n)}|;\ n = 1, 2, \ldots\}$ are not necessarily identical. Again we write

$$Z^{(n)} := \min\left(X_\ell^{(n)},\ \ell = 1, \ldots, |L^{(n)}|\right). \qquad (8)$$

To state the requisite assumptions, we introduce the quantities

$$\lambda_\ell^{(n)} := (m(G_\ell^{(n)}))^{-1}, \qquad \ell = 1, \ldots, |L^{(n)}|, \quad n = 1, 2, \ldots.$$

We assume that the following conditions hold:

Assumption 1 *There exists* $\lambda > 0$ *such that*

$$\lim_{n \to \infty} \sum_{\ell=1}^{|L^{(n)}|} \lambda_\ell^{(n)} = \lambda. \qquad (9)$$

[1] Here, $|A|$ denotes the cardinality of the set A.

Assumption 2 *For every $x \geq 0$,*

$$\lim_{n\to\infty}\left(\max_{\ell=1,\ldots,|L^{(n)}|} G_\ell^{(n)}(x)\right) = 0. \tag{10}$$

A more concrete way to express Assumption 2 is as follows: For every $x \geq 0$ and any given $\varepsilon > 0$, there exists an integer $n^\star = n^\star(x;\varepsilon)$ such that

$$\max_{\ell=1,\ldots,|L^{(n)}|} G_\ell^{(n)}(x) \leq \varepsilon \quad \text{for all } n \geq n^\star. \tag{11}$$

Theorem 3..1 *Under Assumptions 1 and 2, it holds that $Z^{(n)} \Longrightarrow_n \mathcal{E}_\lambda$, i.e.,*

$$\lim_{n\to\infty} \mathbf{P}\left[Z^{(n)} \leq x\right] = \begin{cases} 1-e^{-\lambda x} & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases} . \tag{12}$$

The so-called homogeneous case assumes the existence of i.i.d. rvs $\{X_k,\ k = 1,2,\ldots\}$ distributed according to some distribution F given by (5) with another distribution G on $\mathbb{R}_+$ ($m(G) = 1/\lambda$), and takes

$$X_\ell^{(n)} = |L^{(n)}| \cdot X_\ell, \quad \ell = 1,\ldots,|L^{(n)}|$$

for each $n = 1,2,\ldots$, with corresponding distributions

$$G_\ell^{(n)}(x) = G(\frac{x}{|L^{(n)}|}), \quad x \geq 0,\ \ell = 1,\ldots,|L^{(n)}| \ .$$

Assumption 2 now reads $\lim_{n\to\infty} G_\ell^{(n)}(x) = G(0) = 0$, and the convergence (12) reads

$$|L^{(n)}| \cdot \min\left(X_\ell\ ;\ \ell = 1,\ldots,|L^{(n)}|\right) \Longrightarrow_n \mathcal{E}_\lambda \ .$$

Assumption 1 is automatically satisfied since $(m(G_\ell^{(n)}))^{-1} = \frac{\lambda}{|L^{(n)}|}$ for each $\ell = 1,\ldots,|L^{(n)}|$.

Proof: This result is a simple application of Palm's Theorem [5, Thm. 5-14, p. 157]: Fix $n = 1,2,\ldots,$ and consider $|L^{(n)}|$ independent *delayed* renewal processes $\{N_\ell^{(n)}(t), t \geq 0\}$, $\ell = 1,\ldots,|L^{(n)}|$, which are specified as follows. For each $\ell = 1,\ldots,|L^{(n)}|$, the epoch $A_\ell^{(n)}$ of the first arrival is distributed according to $F_\ell^{(n)}$, so that

$$A_\ell^{(n)} =_{st} X_\ell^{(n)}. \tag{13}$$

Next, the time between the first and second renewal epochs of the renewal processes $\{N_\ell^{(n)}(t), t \geq 0\}$ is distributed according to $G_\ell^{(n)}$. Consequently, the process $\{N_\ell^{(n)}(t), t \geq 0\}$ is a renewal process in *equilibrium*.

The aggregate point process $\{N^{(n)}(t), t \geq 0\}$ is defined by

$$N^{(n)}(t) = \sum_{\ell=1}^{|L^{(n)}|} N_\ell^{(n)}(t), \quad t \geq 0.$$

We readily see that the first event in this aggregate point process takes place at a time given by

$$\inf\{t \geq 0 \mid N^{(n)}(t) \geq 1\} = \min_{\ell=1,\ldots,|L^{(n)}|} A_\ell^{(n)}.$$

By Palm's Theorem [5], the rv $\min_{\ell=1,\ldots,|L^{(n)}|} A_\ell^{(n)}$ converges in distribution to an exponential rv with parameter λ. Consequently, upon using (13), we find

$$\lim_{n\to\infty} \mathbf{P}\left[\min_{\ell=1,\ldots,|L^{(n)}|} X_\ell^{(n)} \leq x\right] = \begin{cases} 1-e^{-\lambda x} & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases} .$$

This completes the proof. ■

The results indicate that when the number of hops is large, the distribution of path duration can be accurately approximated by an exponential rv under a set of mild conditions. In fact, in the i.i.d. case the only assumption required for convergence is that the link duration satisfies $G(0) = 0$. In addition, as noted earlier, (5) tells us that the PDF of the duration of an one-hop path is a non-increasing function. This observation contrasts with the numerical results (Figure 6) in [8], where the authors suggest, based on simulation results, that the one-hop path duration may not have a non-increasing PDF. We suspect that this might be due to (i) the limited number of statistics they collected from the simulation as a result of low mobility or to (ii) the slightly different definition of path duration used in the paper. Note that the PDF plots become much smoother with increasing mobility or speed of nodes in [8], thereby yielding a larger number of collected samples (*e.g.,* Figure 6 and 7 vs. Figure 8 - 10 in [8]).

4. A Simple Model for Distribution of Link and Path Durations Under the Random Waypoint Mobility Model

In this section we first develop a simple model for computing the distribution of link durations under the Random Waypoint (RW) mobility model without pause. We then use this model to compute the distribution of link and path durations with a varying number of hops. We show through a numerical example that the distribution of a path duration does indeed converge to an exponential distribution with increasing hop count.

4.1. Fixed Area Case

Let us first present the model used for computing the link and path duration distributions. Suppose that there are N mobile nodes. Initially these nodes are placed on a disk of radius a according to a uniform distribution independently of each other. Without loss of generality we assume that this disk is centered at the origin. Although here we use a disk, a similar model with a rectangular region can be used as well [1]. The transmission range r_{min} of the nodes is assumed to be the same for all nodes.

Each node selects a destination uniformly on the disk, and then, independently of the destination selected, chooses the speed at which it will move towards the destination along a straight line connecting the current location and the chosen destination. When it reaches the destination, the node repeats this procedure. We assume that the nodes have been repeating the above many times so that the system has reached steady state. From the circular symmetry, this stationary spatial distribution depends only on the distance from the origin. The spatial distribution of a node at steady state on a line segment [0, a] is given in [1], and can be well approximated by

$$\phi_a(r) = \begin{cases} \frac{2}{\pi \cdot a^2}\left(1 - \frac{r^2}{a^2}\right) , & 0 \le r \le a \\ 0 , & \text{otherwise} \end{cases} \quad (14)$$

In other words, (14) satisfies the relationship

$$\int_0^{2\pi}\int_0^a r \cdot \phi_a(r) \cdot dr \cdot d\theta = 1$$

and the PDF of the position of a node in polar coordinate is given by

$$f_{\mathbf{r},\Theta}(r,\theta) = \frac{2}{\pi \cdot a^2}\left(r - \frac{r^3}{a^2}\right) = r \cdot \phi_a(r) . \quad (15)$$

Clearly, due to the boundary effects the spatial distribution of a node is not uniform on the disk and is decreasing in the distance from the center of the disk.

4.2. Asymptotic Case

In this subsection we argue that, although (14) is not uniform for any fixed radius a, for sufficiently large a, given that node j is in the transmission range of another node i, *i.e.*, nodes i and j are neighbors of each other at some time $t \ge 0$, the conditional distribution of the position of node j can be approximated by a uniform distribution on the disk centered at the position of node i with radius r_{min}.

Let N be the number of nodes in the system, and initially each of these nodes is placed on a disk of radius a_N according to a uniform distribution on the disk independently of other nodes. Here the radius a_N depends on the number of nodes in the system, and in order to keep the density of the nodes, *i.e.*, the number of nodes per unit area, constant with varying N, we assume that the radius is scaled according to $a_N = \alpha \cdot \sqrt{N}$ for some positive constant α. The transmission range r_{min} of the nodes is assumed to be constant irrespective of N. We denote by D_N the disk centered at the origin with radius a_N. We assume that the nodes have repeated the procedure described in the previous subsection many times and the system has reached steady state. For a fixed N, in order to explicitly indicate the dependency on N the probability of an event A is denoted by $\mathbf{P}_N[A]$.

Denote the position of node i by $\mathbf{X}_i = (\nu_i, \eta_i)$ and let $D(\mathbf{X}_i)$ be the disk centered at $\mathbf{X}_i$ with radius r_{min}. We will demonstrate that, given that node j is a neighbor of node i, the conditional distribution of the position $\mathbf{X}_j = (\nu_j, \eta_j)$ of node j can be well approximated by a uniform distribution on the disk $D(\mathbf{X}_i)$ as a_N becomes large.

For a fixed N, let

$$\mathbf{P}_N[A] := \int_A f_{a_N}(\mathbf{X})\, d\mathbf{X} \ \text{ for all } A \subset \mathbb{R}^2$$

where

$$f_{a_N}(\mathbf{X}) = \begin{cases} \phi_{a_N}(||\mathbf{X}||) , & \text{if } ||\mathbf{X}|| \le a_N \\ 0 , & \text{otherwise} \end{cases} ,$$

and $||\mathbf{X}|| = \sqrt{\nu^2 + \eta^2}$. If the value of $\mathbf{X}_i \in D_N$ is known, the conditional PDF of $\mathbf{X}_j$ given $\{\mathbf{X}_j \in D(\mathbf{X}_i)\}$ is given by

$$f^N_{\mathbf{X}_i}(\mathbf{X}_j) = \begin{cases} \frac{f_{a_N}(\mathbf{X}_j)}{\mathbf{P}_N[D(\mathbf{X}_i)]} , & \mathbf{X}_j \in D_N \cap D(\mathbf{X}_i) \\ 0 , & \text{otherwise} \end{cases} \quad (16)$$

so that for any $B \subset \mathbb{R}^2$, the conditional probability of $\mathbf{X}_j$ belonging to B is given by

$$\mathbf{P}_N[\mathbf{X}_j \in B \mid \mathbf{X}_j \in D(\mathbf{X}_i)] = \int_{B \cap D(\mathbf{X}_i)} f^N_{\mathbf{X}_i}(\mathbf{X}_j)\, d\mathbf{X}_j . \quad (17)$$

For any $\epsilon > 0$, define

$$A^\epsilon_N = \Big\{\mathbf{X} \in D_N \mid |f^N_{\mathbf{X}}(\mathbf{X}_1) - f^N_{\mathbf{X}}(\mathbf{X}_2)| \le \epsilon \text{ for all } \mathbf{X}_1, \mathbf{X}_2 \in D(\mathbf{X})\Big\} .$$

Note that if $\mathbf{X}_i \in A^\epsilon_N$, then $|\, f^N_{\mathbf{X}_i}(\mathbf{X}_j) - 1/(\pi r^2_{min})\, | \le \epsilon$ for all $\mathbf{X}_j \in D(\mathbf{X}_i)$. In other words, for all $\mathbf{X}_i \in A^\epsilon_N$, the deviation of the conditional PDF $f^N_{\mathbf{X}_i}(\mathbf{X}_j)$ from a uniform distribution is upper bounded by ϵ. The following proposition now tells us that as a_N increases, the probability that $\mathbf{X}_i$ will belong to A^ϵ_N at steady state goes to one for *all* $\epsilon > 0$. Since ϵ can be selected arbitrarily small, it implies that for sufficiently large N the conditional distribution of $\mathbf{X}_j$ given $\{\mathbf{X}_j \in D(\mathbf{X}_i)\}$ can be approximated by a uniform distribution.

Proposition 1 *For* all $\epsilon > 0$*, we have*

$$\lim_{N\to\infty} \mathbf{P}_N[A^\epsilon_N] = 1 .$$

Proof: Suppose that $\beta_N, N \ge 1$, is a sequence of positive constants in (0, 1) such that (i) $\lim_{N\to\infty}\beta_N = 0$, and (ii) $\lim_{N\to\infty}\beta_N \cdot a_N = \infty$. Let $D^{'}_N$ be the disk centered at the origin with radius $(1-\beta_N)a_N$. Then, from (14) we have $\mathbf{P}_N\left[D^{'}_N\right] \ge (1-\beta_N)^2$, and

$$\lim_{N\to\infty} \mathbf{P}_N\left[D^{'}_N\right] = 1 . \quad (18)$$

For each N, we denote the conditional PDF of $\mathbf{X}_j$ given $\mathbf{X}_j \in D(\mathbf{X})$ for some $\mathbf{X} \in D^{'}_N$ by $f^N_{\mathbf{X}}(\mathbf{X}_j)$, and the disk centered at $\mathbf{X}$ with radius r_{min} by $D(\mathbf{X})$.

In order to prove the proposition we now show that

$$\limsup_{N\to\infty} \sup_{\mathbf{X}\in D^{'}_N} \sup_{\mathbf{X}_1,\mathbf{X}_2\in D(\mathbf{X})} \left|f^N_{\mathbf{X}}(\mathbf{X}_1) - f^N_{\mathbf{X}}(\mathbf{X}_2)\right| = 0 . \quad (19)$$

First, from the circular symmetry and (14), one can show that the inner supremum in (19) is achieved by two points on $D(\mathbf{X})$ that are either the closest or farthest from the origin. Second, from the monotonicity and concavity of (14) over $[0, a]$ and the definition of conditional PDF in (16), the outer supremum is achieved by the boundary points of D'_N. Thus, we have

$$\begin{aligned}
&\sup_{\mathbf{X}\in D'_N} \sup_{\mathbf{X}_1,\mathbf{X}_2\in D(\mathbf{X})} \left| f^N_{\mathbf{X}}(\mathbf{X}_1) - f^N_{\mathbf{X}}(\mathbf{X}_2)\right| \\
&= \sup_{\mathbf{X}_1,\mathbf{X}_2\in D(\mathbf{X}^N_\star)} \left| f^N_{\mathbf{X}^N_\star}(\mathbf{X}_1) - f^N_{\mathbf{X}^N_\star}(\mathbf{X}_2)\right| \\
&= f^N_{\mathbf{X}^N_\star}((1-\beta_N)a_N - r_{min}, 0) \\
&\qquad - f^N_{\mathbf{X}^N_\star}((1-\beta_N)a_N + r_{min}, 0) \qquad (20)
\end{aligned}$$

where $\mathbf{X}^N_\star = ((1-\beta_N)a_N, 0)$. Let $\beta^c_N = 1 - \beta_N$.

$$\begin{aligned}
&(20) \\
&= \frac{f_{a_N}(\beta^c_N \cdot a_N - r_{min}, 0) - f_{a_N}(\beta^c_N \cdot a_N + r_{min}, 0)}{\mathbf{P}_N\left[D(\mathbf{X}^N_\star)\right]} \\
&\le \frac{f_{a_N}(\beta^c_N \cdot a_N - r_{min}, 0) - f_{a_N}(\beta^c_N \cdot a_N + r_{min}, 0)}{\pi r^2_{min} \cdot \inf_{\mathbf{X}\in D(\mathbf{X}^N_\star)} f_{a_N}(\mathbf{X})} \\
&= \frac{f_{a_N}(\beta^c_N \cdot a_N - r_{min}, 0) - f_{a_N}(\beta^c_N \cdot a_N + r_{min}, 0)}{\pi r^2_{min} \cdot f_{a_N}(\beta^c_N \cdot a_N + r_{min}, 0)} \\
&= \frac{\frac{2}{\pi \cdot a_N^2}\left(1 - \frac{(\beta^c_N \cdot a_N - r_{min})^2}{a_N^2}\right)}{\pi r^2_{min} \cdot \frac{2}{\pi \cdot a_N^2}\left(1 - \frac{(\beta^c_N \cdot a_N + r_{min})^2}{a_N^2}\right)} \\
&\quad - \frac{\frac{2}{\pi \cdot a_N^2}\left(1 - \frac{(\beta^c_N \cdot a_N + r_{min})^2}{a_N^2}\right)}{\pi r^2_{min} \cdot \frac{2}{\pi \cdot a_N^2}\left(1 - \frac{(\beta^c_N \cdot a_N + r_{min})^2}{a_N^2}\right)} \\
&= \frac{4\beta^c_N r_{min}}{\pi r^2_{min}\left(2\beta_N \cdot a_N - \beta_N^2 \cdot a_N - 2\beta^c_N r_{min} - \frac{r^2_{min}}{a_N}\right)} \\
&= \frac{4\beta^c_N r_{min}}{\pi r^2_{min}(\beta_N \cdot a_N(2-\beta_N) - \Gamma)} ,
\end{aligned}$$

where

$$\Gamma = 2\beta^c_N r_{min} + \frac{r^2_{min}}{a_N} .$$

Therefore, we have

$$\begin{aligned}
&\limsup_{N\to\infty} \sup_{\mathbf{X}\in D'_N} \sup_{\mathbf{X}_1,\mathbf{X}_2\in D(\mathbf{X})} \left| f^N_{\mathbf{X}}(\mathbf{X}_1) - f^N_{\mathbf{X}}(\mathbf{X}_2)\right| \\
&\le \limsup_{N\to\infty} \frac{4\beta^c_N r_{min}}{\pi r^2_{min}(\beta_N \cdot a_N(2-\beta_N) - \Gamma)} \\
&= 0 , \qquad (21)
\end{aligned}$$

where the equality follows from the assumption that $\beta_N \cdot a_n \to \infty$ and $\beta_N \to 0$ as $N \to \infty$. Now, the proposition follows from (18) and (21). ■

4.3. Distribution of Link and Path Duration

In this subsection, using the model described in the previous subsections, we discuss how to approximate the distribution of link durations under the RW mobility model without pause. To do so, we focus on two nodes, denoted by n_1 and n_2, that become neighbors at some time $t \ge 0$ as shown in Figure 3, and find the distribution of the link duration between them. We assume that the radius of the disk a is sufficiently large $(a \gg r_{min})$ so that the conditional distribution discussed in the previous subsection can be approximated by a uniform distribution. Without loss of generality we assume $a = 1$ and scale other parameters accordingly. This also implies that the average distance between two consecutive destination locations selected by a node, is much larger than the transmission range r_{min}. Therefore, in most cases when two nodes become neighbors, with a high probability, neither of these two nodes will reach its current destination location before the link between them is torn down after they move out of the transmission range of each other. In other words, the average travel time of a node between two consecutive destination locations selected by the node is much larger than the average link duration between two nodes. Hence, for the simplicity of analysis we assume that neither node reaches its destination while they are neighbors, and truncate the link duration to model the arrival of the nodes at the destination locations. Rather than modeling the mobility of both nodes explicitly, we only model the net effects of mobility between the nodes by pretending that node n_1 is fixed and modeling the relative motion of node n_2 with respect to n_1. This will be discussed in more details shortly.

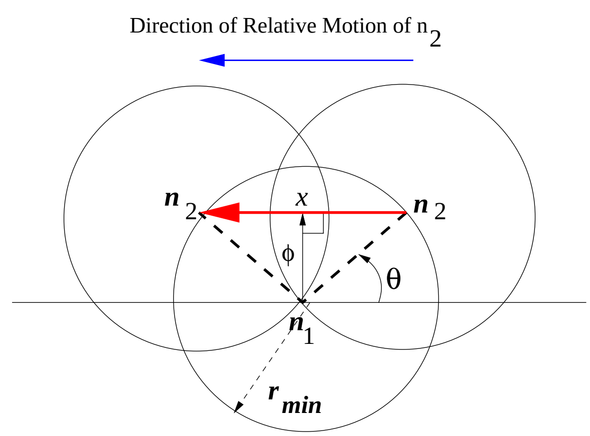


Figure 3: Link duration.

Let ϕ denote the distance between n_1 and n_2 when they are closest to each other. The location of node n_2 when the distance between them is ϕ, is denoted by x as shown in Figure 3. We draw a reference line which is perpendicular to the arrow from n_1 to x and goes through n_1, which is shown as a long solid line in Figure 3. Note that the relative motion of node n_2 with respect to node n_1 is parallel to the reference line. Given that the relative motion of node n_2 is parallel to this reference line and they become neighbors at some point (before the angle between the reference line and the arrow from n_1 to n_2 becomes $\frac{\pi}{2}$), under the steady state assumption along with the assumption $r_{min} \ll 1$, the minimum distance ϕ is approximately uniformly distributed on the interval $[0, r_{min})$ from Proposition 1. Given the (relative) speed of the node n_2, denoted by S, and the angle of the arrow from node n_1 to node n_2 makes with the reference line when n_2 first comes within the transmission range of n_1, denoted by Θ in Figure 3, the duration of the link between these two nodes is given by

$$D(S,\Theta) = \frac{2r_{min}\cos\Theta}{S}\,. \qquad (22)$$

Therefore, from the independence of mobility of nodes, we can approximate the CDF of link duration D conditional on the relative speed S as follows.

$$\begin{aligned}
&\mathbf{P}\left[D \le d \mid S = s\right] \\
&= \mathbf{P}\left[\frac{2r_{min}\cos\Theta}{s} \le d\right] \\
&= \mathbf{P}\left[\cos\Theta \le \frac{d\cdot s}{2r_{min}}\right] \\
&= \mathbf{P}\left[\sin\Theta \ge \sqrt{1-\left(\frac{d\cdot s}{2r_{min}}\right)^2}\right] \qquad (23)\\
&\approx \begin{cases} \left(1-\sqrt{1-\left(\frac{d\cdot s}{2r_{min}}\right)^2}\right)\cdot u(d) & \text{if } d \le \frac{2r_{min}}{s} \\ 1 & \text{if } d > \frac{2r_{min}}{s} \end{cases} \\
&:= g(d,s)
\end{aligned}$$

where $u(\cdot)$ is a unit step function.[2] The first equality follows from (22), and the last approximation follows from the fact that $r_{min}\cdot\sin\Theta$ is the minimum distance ϕ between the nodes and hence $\sin\Theta$ is approximately uniformly distributed in $[0,1)$ as explained earlier. Therefore, $\mathbf{P}\left[D \le d\right]$ can be approximated using (23), and we have

$$\begin{aligned}
\mathbf{P}\left[D \le d\right] &= \int_s \mathbf{P}\left[D \le d \mid S = s\right] d\mathcal{H}(s) \\
&\approx \int_s g(d,s)\, d\mathcal{H}(s) \qquad (24)
\end{aligned}$$

where $\mathcal{H}$ is the distribution of the relative speed S.

We now turn to the distribution $\mathcal{H}$ of S. Since the mobility of a node is independent of that of the others, the relative motion of node n_2 from the perspective of node n_1 is simply the difference $V = V^{(2)} - V^{(1)}$, where the vector $V^{(i)}$ represents the velocity of the node n_i ($i = 1,2$). This is shown in Figure 4.

The relative speed S is given by $\sqrt{V_1^2 + V_2^2}$, where V_1 and V_2 are the vertical and horizontal components of V, respectively, in Figure 4. From the definition, the CDF of S is given by

$$\mathbf{P}\left[S \le s\right] = \mathbf{P}\left[V_1^2 + V_2^2 \le s^2\right]\,. \qquad (25)$$

One can simplify (25) as follows. For the simplicity of analysis we assume that the angle $\Theta^{'}$ between $V^{(2)}$ and $-V^{(1)}$ is uniformly distributed in [0, 2π). Numerical examples obtained using ns-2 simulation [4] show that this assumption introduces only a negligible amount of discrepancy in link duration distribution. The speed of each node, given by $S^{(i)}, i = 1,2$, does not depend on the angle of its motion. Hence, the relative speed S between the nodes can be approximated by

$$S = \sqrt{(S^{(1)} + S^{(2)}\cdot\cos\Theta')^2 + (S^{(2)}\cdot\sin\Theta')^2}\,,$$

and, for all $s \ge 0$,

$$\begin{aligned}
&\mathbf{P}\left[S \le s\right] \qquad (26)\\
&= \mathbf{P}\left[(S^{(1)} + S^{(2)}\cdot\cos\Theta^{'})^2 + (S^{(2)}\cdot\sin\Theta^{'})^2 \le s^2\right],
\end{aligned}$$

Therefore, if we know the (stationary) distribution of $S^{(i)}$, we can numerically compute the CDF of S from (26).

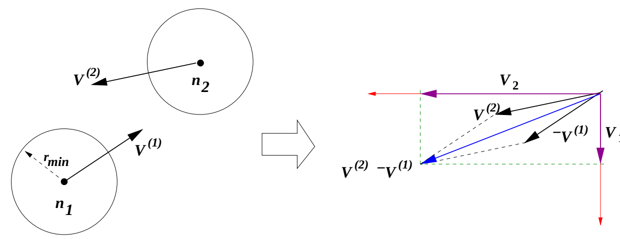


Figure 4: Relative motion of n_2 with respect to n_1.

Consider the RW model where the speed is selected uniformly from (s_{min}, s_{max}), where s_{max} and s_{min} are the maximum and minimum speed of a node. Then, the PDF of the stationary distribution of node i's speed, $S^{(i)}$, under the above RW model is given by the following [6]:

$$f(s) = \frac{1}{s(\ln s_{max} - \ln s_{min})}\,. \qquad (27)$$

Hence, from (26) and (27) one can numerically compute the distribution $\mathcal{H}$ of S.

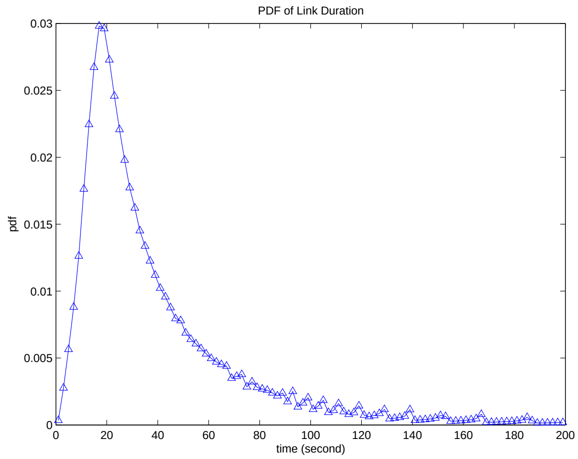
From the distribution of S the distribution of a link duration D can be computed using (24). Using the same example above where the speed of a node is randomly selected from the bounded interval $(s_{min},\ s_{max})$, we compute the distribution of D with the transmission range $r_{min} = 250$ m. The PDF of a link duration D truncated at 200 seconds is shown in Figure 5(a). The PDF in Figure 5(a) is very similar to the experimental results provided in [8] (see Figure 3 in [8]), thereby confirming the accuracy of the model presented in this section.

Now using the link duration distribution obtained above we can calculate the distribution of an excess life F from (5) and that of a path duration with n hops from

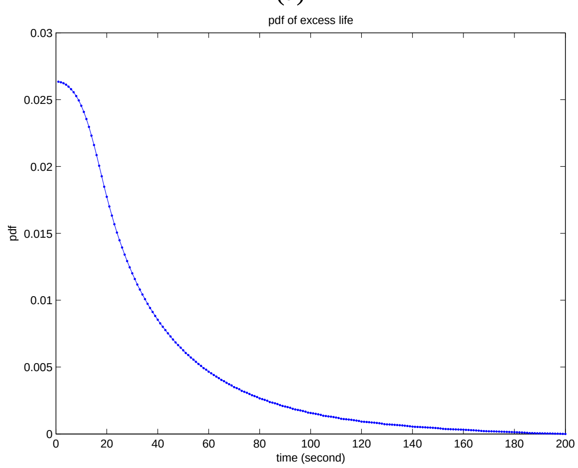
$$f_{Z^{(n)}}(x) = \frac{n}{m(G)}(1-F(x))^{n-1}(1-G(x)), \quad x \ge 0,$$

where G is the link duration distribution, *i.e.*, $G(d) = \mathbf{P}\left[D \le d\right]$. These are plotted in Figures 5(b) and 6, respectively, for the example in Figure 5(a). As one can see, the PDF of the excess life is non-increasing and is quite different from that of a link duration. Therefore, in

[2] Here $\mathbf{P}\left[D \le d \mid S = s\right]$ denotes $\lim_{\epsilon\downarrow 0}\mathbf{P}\left[D \le d \mid s < S \le s+\epsilon\right]$.

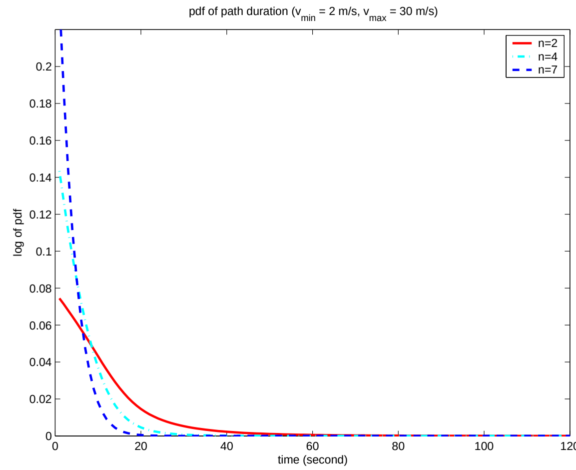


(a)

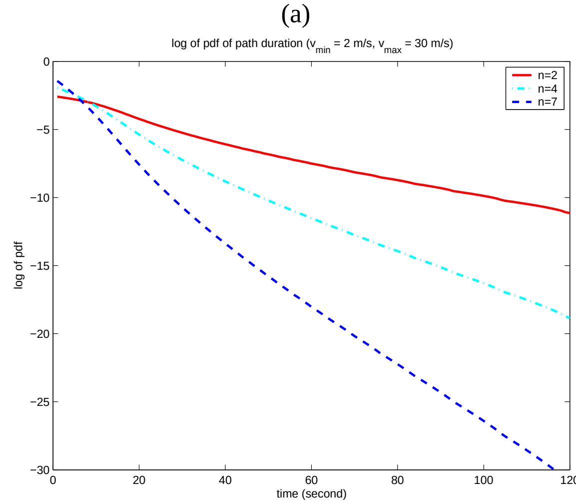


(b)

Figure 5: The pdf of (a) link duration and (b) excess life ($s_{min} = 2$ m/s, $s_{max} = 30$ m/s, and $r_{min} = 250$ m).



(a)



(b)

Figure 6: Plot of pdf and log of pdf of a path duration with $s_{min} = 2$ m/s, $s_{max} = 30$ m/s, and $r_{min} = 250$ m.

order to accurately model the path duration the excess life needs to be used rather than the duration of a link. Figure 6 shows the distribution of a path duration with hop counts $n = 2$, 4, and 7. It is easy to see that with the increasing number of hops n, the distribution of the path duration increasingly resembles an exponential distribution, validating our analytical results.

5. Correlation Between Neighboring Links

In this section we take a look at the independence assumption of the excess lives of the links along a path and attempt to provide some justification for the assumption. If two links along a path are separated by at least one link in-between, since no nodes are shared by the links, the excess lives of these links are expected to be at most weakly dependent, if not independent. However, two neighboring links share a node. In the example shown in Figure 7, node n_2 is shared by links (n_1, n_2) and (n_2, n_3). Since the excess lives of these two links depend on the mobility of node n_2, the independence assumption is clearly not true in general in this case, and requires some justification. Although, strictly speaking the independence assumption does not hold between two neighboring links, we demonstrate that the correlation coefficient between the excess lives of two neighboring links, which is a measure of dependency between them, is nontheless rather small.

Without loss of generality we assume that the neighboring links are given by (n_1, n_2) and (n_2, n_3), which we denote by ℓ_1 and ℓ_2, respectively, and that packets traverse nodes in the order $n_1 \rightarrow n_2 \rightarrow n_3$. Here we assume that the underlying routing protocol does not attempt to optimize the selected path, and nodes n_1 and n_3 are uniformly distributed within the transmission range of node n_2 at the time of path selection. A more efficient routing protocol should not select n_2 as the next hop from n_1 if n_3 is in the transmission range of n_1. However, due to the broadcast nature of a path request packet, node n_3 may not receive the request message correctly from node n_1, and may reply only to the broadcast message by node n_2. We denote the excess life of link ℓ_i by X_i $(i = 1, 2)$. The correlation coefficient of X_1 and X_2 is defined [10] to be

$$\rho_{X_1,X_2} = \frac{E[X_1 X_2] - E[X_1]E[X_2]}{\sqrt{Var(X_1)Var(X_2)}}$$

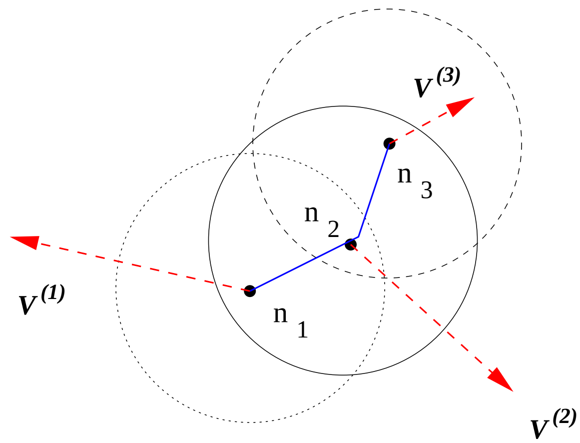


Figure 7: Neighboring links.

$$= \frac{E[X_1X_2] - E[X_1]^2}{Var(X_1)}.$$

From the previous section we know how to compute the distribution of the excess life, and hence we can compute $E[X_1]$ and $Var(X_1) = E[(X_1)^2] - E[X_1]^2$. The correlation $E[X_1X_2]$ can be computed by conditioning on the speed of node n_2, denoted by $S^{(2)}$, as follows:

$$E[X_1X_2] = E\left[E\left[X_1X_2 \mid S^{(2)}\right]\right] = \int_s E\left[X_1X_2 \mid S^{(2)} = s\right] d\mathcal{F}(s)$$

where $\mathcal{F}$ is the distribution of $S^{(2)}$ at steady state. If we adopt the model in Section 4 and assume that the speed of nodes is selected uniformly from $(s_{min},\ s_{max})$, then from (27) we find

$$E[X_1X_2] = \int_{s_{min}}^{s_{max}} E\left[X_1X_2 \mid S^{(2)} = s\right] \frac{1}{s(\ln s_{max} - \ln s_{min})} ds \quad (28)$$

We now describe how to compute the conditional expected value $E\left[X_1X_2 \mid S^{(2)} = s\right]$. This requires computing the conditional distribution of the relative speed between n_2 and its neighbors given the value of $S^{(2)}$, which is different from the a priori distribution of the relative speed used in Section 4. However, the calculation of this conditional distribution can be carried out in a similar manner. Without loss of generality we take the link $l_1 = (n_1, n_2)$ to compute the link duration distribution conditional on the value of $S^{(2)}$. Suppose that the mobility of node $n_i, i = 1, 2$, is represented by $V^{(i)}$ as before. The angle $\Theta^{'}$ between $V^{(1)}$ and $-V^{(2)}$ is assumed to be uniformly distributed in $[0,\ 2\pi)$. Hence, with a little abuse of notation, the CDF of the relative speed S of node n_1 with respect to node n_2 conditional on $S^{(2)}$ is given by

$$\begin{aligned}&\mathbf{P}\left[S \le s \mid S^{(2)} = s_2\right]\\ &= \mathbf{P}\left[(S^{(1)} \cdot \cos\Theta^{'} + s_2)^2 + (S^{(1)} \cdot \sin\Theta^{'})^2 \le s^2\right]\\ &:= \mathcal{F}_{s_2}(s)\end{aligned} \quad (29)$$

where $S^{(1)}$ is the speed of node n_1, and $\Theta^{'} \sim Unif[0,\ 2\pi)$. Once the conditional distribution of the relative speed in (29) is obtained, we can compute the conditional distribution of link duration from

$$\begin{aligned}&\mathbf{P}\left[D \le d \mid S^{(2)} = s_2\right]\\ &= \int_s \mathbf{P}\left[D \le d \mid S^{(2)} = s_2, S = s\right] d\mathcal{F}_{s_2}(s)\\ &= \int_s \mathbf{P}\left[D \le d \mid S = s\right] d\mathcal{F}_{s_2}(s)\\ &:= G_{s_2}(d)\end{aligned} \quad (30)$$

where $\mathcal{F}_{s_2}$ is the conditional distribution in (29). Using (30), the conditional excess life distribution given $S^{(2)} = s_2$ can be computed from (5) as

$$\begin{aligned}&F(x \mid S^{(2)} = s_2)\\ &= \begin{cases} \frac{1}{m(G_{s_2})}\int_0^x (1 - G_{s_2}(y))dy & \text{if } x > 0\\ 0 & \text{if } x \le 0\end{cases}\\ &:= F_{s_2}(x)\end{aligned}$$

where $m(G_{s_2})$ is the mean of G_{s_2}.

Since X_1 and X_2 are conditionally independent given the speed of node n_2, the correlation $E[X_1X_2]$ can be computed as

$$\begin{aligned}&E[X_1X_2]\\ &= E\left[E\left[X_1X_2 \mid S^{(2)}\right]\right]\\ &= \int_{s_2} E\left[X_1X_2 \mid S^{(2)} = s_2\right] d\mathcal{F}(s_2)\\ &= \int_{s_2}\left(\int_{x_1} x_1 dF_{s_2}(x_1)\int_{x_2} x_2 dF_{s_2}(x_2)\right) d\mathcal{F}(s_2)\end{aligned} \quad (31)$$

The correlation coefficient ρ of X_1 and X_2 in the example used earlier where the node speed is randomly selected from $(2, 30)$ m/s can be computed from (31) and is approximately 0.0305. Hence, the correlation between the neighboring links is rather weak, although they are not independent.

6. Conclusions

We have studied the distributional properties of path duration in multi-hop wireless networks. We have shown that, under a set of mild conditions, the distribution of path duration converges to an exponential distribution with appropriate scaling as the number of hops increases. We have verified our results using a simple model for RW mobility model. Our model allows us to compute the distribution of a link duration given the distribution of speed of a node and, hence, the distribution of excess life and path duration.

For our analysis we have assumed that the link durations are mutually independent. Although this may not be strictly true in general, we have shown that in the case of RW mobility model the correlation of the excess life between two neighboring links along a path is rather weak. We are currently investigating the correlation structure of the link durations to understand the implications of dependency of link durations on the on-demand routing protocols and distributional properties of a path duration.

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