GLOBAL STABILITY WITH A STATE-DEPENDENT DELAY IN RATE CONTROL

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Abstract: We characterize the stability conditions with an *arbitrary* state-dependent time-varying round-trip delay in the context of single-flow and single-resource for Kelly's optimization based rate control scheme using invariance techniques. It is shown that the stability conditions with an arbitrary *fixed* delay are sufficient for the stability of the system with a state-dependent time-varying delay.

1. INTRODUCTION

With the advent of cheap internetworking technology, it has become possible to connect traditional control devices over a network. While networks provide the flexibility of connecting multiple equipments and sharing the bandwidth, by their very design, they can transfer only delayed information. In particular, this delay can be time-varying. Although the delay is generally tolerable for data transfer, from stability perspective, it can create dynamical instability even for otherwise stable linear systems. Applications such as remotely operated robots for deep space exploration or tiny medical imaging devices over a wireless network have to face the reality of time-varying delays. Considering the broad class of applications which are either already operating over a network or will be doing so, it is important to examine the natural settings for systems over a network with time-varying delays. This paper explores the first step towards this problem, which is an examination of the time-varying delay in the network and its effect on network resource allocation algorithms.

Kelly (Kelly, 1997) has suggested that the problem of rate allocation for elastic traffic can be posed as one of achieving maximum aggregate utility of the users and proposed an optimization framework for rate allocation in the Internet. Here the utility of a user could either represent the true utility or preferences of the user or a utility function that is assigned to the user by the end user rate control algorithm, *e.g.,* Transmission Control Protocol (TCP). In the latter case the selection of the utility function determines the end user algorithm and the desired operating point. Using the proposed framework he has shown that the system optimum is achieved at the equilibrium between the end users and resources. Based on this observation researchers have proposed various rate-based algorithms, in conjunction with a variety of active queue management (AQM) mechanisms, that solve the system optimization problem or its relaxation (Kelly, 1997; Low and Lapsley, 1997).

Initially the convergence of these algorithms (Kelly et al., 1998), however, has been established only in the absence of feedback delay, and the implications of feedback delay have been left open. Accurate modeling of the communication delay is especially important when the delay is non-negligible or widely varying with uncertainty. A good example of such an environment is multi-hop mobile wireless networks.

There has been some work on establishing the stability of the system with delays (Alpcan and Basar, 2003; Deb and Srikant, 2002; Johari and Tan, 2000; Mazenc and Niculescu, 2003; Ranjan et al., 2003). However, these models assume fixed delays, where the round-trip delays of packets are fixed regardless of the rates of the users. In other words, the queueing dynamics at the resources have been ignored. It is worth noting that state-dependent delay can change the stability property in a significant way (Büger and Martin, 2002). [1] For instance, the feedback delay can increase without a bound if an infinite buffer size is allowed, which is known as an escape to infinity problem in (Büger and Martin, 2002). is an important revisit our earlier work in a simple single-flow, characterized the stability delay.

We show that even with queue dynamics, the stability conditions for the fixed delay case provided in (Ranjan et al., 2003) are sufficient for the stability. In other words, the detailed dynamics of the round-trip delay do not change the stability conditions in a significant way when one is interested in maintaining stability with an *arbitrary* delay. This is illustrated using a family of popular utility and price functions.

The main contribution of this paper is two folds; first, it provides a robust means of ensuring the system stability in the presence of arbitrary state-dependent delay and gain. Second, we show that a delay differential equation can be studied as a discrete time map without losing dynamical stability properties under certain conditions. This provides a more efficient way of studying the stability property of a system given by a delay differential equation.

Section 2 describes the model for a simple single-flow, single-resource case and presents the (general) stability conditions. We conclude in Section 3.

2. STABILITY CONDITION: SINGLE-FLOW, SINGLE-RESOURCE WITH STATE DEPENDENT DELAY

In this section we consider a flow traversing a single resource. The rate control problem in the optimization framework proposed in (Kelly, 1997) in this simple case is given by the following simple optimization problem:

$$\max_{x \geq 0} U(x) - \int_0^x p(y)dy \qquad (1)$$
$$\text{s. t.} \qquad x \leq C$$

[1] We say that the delay is state-dependent if it depends on the queue size of the resource(s) in our context.

where x is the rate, $U(x)$ is the utility of the user when it receives a rate of x, $p(x)$ is the price per unit flow charged by the resource when the rate is x, and C is the capacity of the resource. The proposed end user algorithm in the absence of delay is given by the following differential equation (Kelly et al., 1998).

$$\frac{d}{dt}x(t) = \kappa\left(w(t) - x(t)\mu(t)\right) \qquad (2)$$

where $w(t)$ is the price per unit time the user is willing to pay, $\mu(t) = p(x(t))$, and $\kappa, \kappa > 0$, is a gain parameter. We will study the stability of the system in (2) with general utility and resource price functions that satisfy a set of assumptions to be stated shortly.

In practice, the rate of a flow is typically limited by the receiver buffer size. Hence, even when there is no congestion the rate of a flow is upper bounded by some constant. Similarly, we assume that there is a lower bound on the rate since the user needs to probe the congestion level of the network by continually transmitting packets, although this lower bound can be arbitrarily small. We denote this lower bound by $X_{min} > 0$. Furthermore, any physical buffer/memory at a resource will be finite no matter how large it may be. Thus, without loss of generality we make the following assumption on the rate of a flow and the buffer size.

Assumption 1. The rate of the flow lies in a compact set $[X_{min},\ X_{max}]$, where $C \leq X_{max} < \infty$. In addition, the buffer size B is finite, *i.e.,* $0 \leq B < \infty$.

Throughout the rest of the paper we implicitly assume that when the rate of a flow reaches the upper bound X_{max} (lower bound X_{min}), the time derivative is given by $\frac{d}{dt}x(t) = \min\{\kappa(w(t) - x(t)\mu(t)), 0\}$ $(\max\{\kappa(w(t) - x(t)\mu(t)), 0\}$ resp.). Also, the assumption places an upper bound on the round-trip delay.

Adopting the end user algorithm given in (Kelly et al., 1998) we assume that $w(t) = x(t) \cdot U^{'}(x(t))$. The congestion signal generated at the resource, *i.e.,* $p(x(t))$, is returned to the user after a time-varying delay. We denote the round-trip delay of the packet whose acknowledgment (ACK) arrives at the source at time t by $\tau_1(t)$. Hence, the rate at which the ACKs or feedback signal arrives at the source at time t is $x(t - \tau_1(t))$. The round-trip delay consists of two components; it has a fixed delay and a time-varying delay that is caused by the time-varying queue size at the resource. We denote the fixed delay by T. Since the fixed delay can lie (i) in the forward path from the source to the receiver for the data packets, (ii) in the reverse path from the receiver to the source for ACKs carrying the feedback information, or (iii) in both forward and reverse paths, there are three different cases that one can consider. If we assume that the fixed delay lies in the reverse path, the system dynamics are given by the following set of delay differential equations:

$$\frac{d}{dt}x(t) = \kappa\Big(x(t)U^{'}(x(t)) \\ -x(t-\tau_1(t))p(x(t-\tau_1(t)))\Big) \quad (3)$$

$$\tau_1(t) = T + \frac{q(t-\tau_1(t))}{C} \quad (4)$$

$$\frac{dq(t)}{dt} = \begin{cases} x(t) - C\ , & \text{if } 0 < q(t) < B \\ [x(t) - C]^+\ , & \text{if } q(t) = 0 \\ [x(t) - C]^-\ , & \text{if } q(t) = B \end{cases} \quad (5)$$

where $q(t)$ is the queue size at the resource at time t, $[a]^+ = \max\{a, 0\}$, and $[a]^- = \min\{a, 0\}$. Here the time-varying delay in fact depends on the state of the system, *i.e.,* queue size.

If the fixed delay lies in the forward path, equations corresponding to (3) - (5) are given by

$$\frac{d}{dt}x(t) = \kappa\Big(xU^{'}(x(t)) \\ -x(t-\tau_1(t))p(x(t-\tau_1(t)+T))\Big)$$

$$\tau_1(t) = T + \frac{q(t-\tau_1(t)+T)}{C}$$

$$\frac{dq(t)}{dt} = \begin{cases} x(t-T) - C\ , & \text{if } 0 < q(t) < B \\ [x(t-T) - C]^+\ , & \text{if } q(t) = 0 \\ [x(t-T) - C]^-\ , & \text{if } q(t) = B \end{cases} \quad (6)$$

Eq. (5) and (6) essentially state that the server is work-conserving, *i.e.,* the server is never idle while the queue is non-empty. Although the evolution of $x(t)$ depends on where the fixed delay lies, in this paper we only consider the case where the fixed delay lies in the reverse path. The other cases can be handled in a similar manner.

Rather than directly dealing with the delay differential equation in (3) we put it in a form that is more amenable to analysis as follows. After normalizing time by T and replacing $t = s \cdot T$, (3) can be rewritten as

$$\frac{1}{T}\frac{d}{ds}x(s) \\ = \kappa\left(x(s)U^{'}(x(s)) - x(s-\tau(s))p(x(s-\tau(s)))\right)$$

$$\nu\frac{d}{ds}x(s) \\ = x(s)U^{'}(x(s)) - x(s-\tau(s))p(x(s-\tau(s)))$$

where $\nu = \frac{1}{T\kappa}$, and $\tau(s) = \frac{\tau_1(t)}{T}$. We normalize the queuing delay to keep the discussion in line with our earlier results for fixed delay cases (Ranjan et al., 2003). We will show that the stability properties of the solutions of (3) can be obtained from an underlying nonlinear difference equation.

In the following subsection we first establish the appropriate setting to derive general convergence results for one dimensional case.

2.1 Convergence Results for State-Dependent Delay

In this subsection we establish the conditions for convergence of the system in (3)-(5) regardless of the communication delay T, gain parameter κ, and the nature of queuing delay. Consider the following substitution:

$$y(t) = x(t)U^{'}(x(t)) := g(x(t)) \\ \text{and } \ f(x(t)) := x(t)p(x(t)). \quad (7)$$

Note that $y(t)$ is simply the price the user is willing to pay at time t based on its rate, and $f(x)$ is the total price charged by the resource when the rate traversing it is x.

We first make the following assumptions on the functions $g(x)$ and $f(x)$.

Assumption 2. (i) The function $g(x)$ is strictly decreasing with $g^{'}(x) < 0$ for all $x > 0$, (ii) the function $f(x)$ is strictly increasing for all $x > 0$, and (iii) $g(x)$ and $f(x)$ are Lipschitz continuous on $[X_{min}\ ,\ X_{max}]$, where X_{min} and X_{max} are the assumed lower and upper bound on the rate, respectively.

Assumption 2(i) implies that the user is rather inelastic, *i.e.,* the user does not react sharply to the changes in resource price. Moreover, it guarantees the existence of the inverse $g^{-1}(\cdot)$. From (7) one can see that a sufficient condition for Assumption 2(ii) is that the resource price function is strictly increasing in the rate traversing the resource.

Assumption 2 allows us the following change of coordinate:

$$x(t) = g^{-1}(y(t)) \Rightarrow \dot{x}(t) = \frac{\dot{y}(t)}{g^{'}(g^{-1}(y(t)))} \quad (8)$$

$$\nu\frac{dy}{dt}(t) \\ = g^{'}(g^{-1}(y(t)))(y(t) - f(g^{-1}(y(t-\tau(t))))) \ .$$

Let $\overline{\kappa}(y(t)) := -g^{'}(g^{-1}(y(t)))$. Clearly, $\overline{\kappa}(y(t)) > 0$ under Assumption 2. Using this substitution in (8) we get the following form.

$$\nu\frac{dy}{dt}(t) \\ = \overline{\kappa}(y(t))\left(f(g^{-1}(y(t-\tau(t)))) - y(t)\right) \quad (9)$$

We study (9) and show that there is a close correspondence between invariance and global stability properties of the discrete time map

$$y_{n+1} = f(g^{-1}(y_n)) := F(y_n) \tag{10}$$

and those of (9). In particular, we will prove that if $y_{n+1} = F(y_n)$ has a stable fixed point, then (9) will have a uniformly constant solution for all possible time delays $T \geq 0$ if the initial function's range is contained in the immediate basin of attraction of this fixed point. The proofs are based on the invariance property of the underlying map $F(\cdot)$ and the monotonicity of function $g(\cdot)$. Note that the map $F(y)$ is strictly decreasing because $g^{-1}(y)$ is strictly decreasing and $f(x)$ is strictly increasing under Assumption 2 and a composition of a strictly increasing function and a strictly decreasing function is a strictly decreasing function.

2.1.1. Existence and Uniqueness of a Solution
Before we provide the convergence results, we comment on the existence and uniqueness of a solution of (3)-(5). To this end we use the framework developed by Hartung and Turi in (Hartung, 1995; Hartung and Turi, 1995). They consider a general setup for delay differential equations with distributed and state-dependent delays:

$$\dot{z}(t) = \xi(t, z(t), \Lambda(t, z_t)) \tag{11}$$

where $\Lambda(\cdot)$ describes the role played by delayed state variable and can be written as

$$\Lambda(t, z_t) = \int_{-r}^{0} d_s\nu(s, t, z_t)z(t+s) \ , \tag{12}$$

where r is the maximum possible delay, $z(t) \in \mathbb{R}^n$ for $n > 0$, z_t denotes the segment $z_t(s) \equiv z(t+s)$ for $s \in [-r, 0]$, $\nu(\cdot, \cdot, \psi)$ is an $n \times n$ matrix valued function of bounded variation on $[-r, 0]$, $\psi \in C([-r, 0], \mathbb{R}^n)$, and the integral is the Stieltjes-integral of $z(t + \cdot)$ with respect to $\nu(\cdot, t, z_t)$. The set $C([-r, 0], \mathbb{R}^n)$ denotes the set of continuous functions on $[-r, 0]$ with domain $\mathbb{R}^n$.

Let $z(t) = [x(t); q(t)]$ and $\dot{z}(t) = [\dot{x}(t); \dot{q}(t)]$. Our model can now be viewed as a special case of (11) and be obtained by extending Example 1.3 in (Hartung, 1995, pp. 2) as follows. Let $\nu(\cdot)$ be a diagonal matrix with

$$\nu_1(s, t, \psi) \equiv \chi_{[-\tau_1(t,\psi),0]}(s) \ , \ \ s \in [-r, 0]$$

where $\tau_1(t, \psi)$ gives the round-trip delay of the user at time t given some continuous function ψ in $\mathbb{R}^2_+$ (in place of z_t), and $\chi_{[-\tau,0]}(s)$ is the characteristic function of the interval $[-\tau, 0]$, *i.e.*,

$$\chi_{[-\tau,0]}(s) = \begin{cases} 1 & \text{if } -\tau \leq s \leq 0 \\ 0 & \text{if } s < -\tau \text{ or } s > 0 \end{cases} ,$$

and $\nu_2(s, t, \psi) \equiv 0$. It is clear that $\nu(\cdot, t, \psi)$ is of bounded variation on $[-r, 0]$ for all $t \in \mathbb{R}_+$. Then, we have

$$\Lambda(t, z_t) = (x(t - \tau_1(t)); 0)$$

The results developed in (Hartung, 1995) tell us that a solution exists if (i) the function $\xi(\cdot)$ belongs to the Banach-space of bounded continuous functions on an appropriate domain of definition, and (ii) the initial function belongs to the space of continuous functions (Hartung, 1995, pp. 15). In our system the function $\xi(\cdot)$ is given by the right-hand side of (3) and (4), and the first condition can be easily verified. The second condition is a reasonable assumption on the initial conditions considered in this work as user rates and queue sizes must be continuous in time. Finally, the uniqueness of a solution can be guaranteed if $\xi(\cdot)$ of (11) is locally Lipschitz in both second and third argument, which can be verified in our case. These conditions provide us the basis for continuation of solutions and studying their stability.

Next we state the assumption for bounded positive solutions.

Assumption 3. Suppose that $I \subset \{x : x \geq X_{min}\}$ is a closed invariant interval under F, *i.e.*, $F(I) \subset I$. In particular, let $I = [a,\ b]$ be compact.

Under this assumption, we have invariance for the solution of (9) for all time $t \geq 0$ and for all $\nu > 0$. Let $d_{max} = 1 + \frac{B}{CT}$, which is the largest possible round-trip delay normalized by T. We denote the set of continuous functions over $[-d_{max}, 0]$ with the range of D by $C([-d_{max}, 0], D)$. Define $X := C([-d_{max}, 0], \mathbb{R}_+)$, and $X_I := \{\phi \in X : \phi(s) \in I \ \forall \ s \in [-d_{max}, 0]\}$. Since the functions involved in (9) are Lipschitz continuous by assumption, a solution exists for all $t \geq 0$ and is unique for any initial function $\phi \in X_I$.[2] Furthermore, the invariance property of the solutions, which is stated below (Theorem 1), ensures that the solutions stay positive and bounded by the initial set they start in, which is assumed to be invariant under map F.

Theorem 1. (Invariance) If $\phi \in X_I$, the corresponding solution $y_\phi(t) = y(t; \phi)$ satisfies $y_\phi(t) \in I$ for all $t \geq 0$. This means that the set I is invariant under (9).

[2] We implicitly assume that the initial function for the queue size satisfies the appropriate conditions discussed earlier and focus on the rate.

Proof: We prove the theorem by contradiction. Suppose that the claim is not true and there exists $\phi \in X_I$ such that $y(t;\phi) \notin I$ for some $t \geq 0$. Let t_0 be the first time when solution $y(t;\phi)$ leaves I, *i.e.,*

$$t_0 = \inf\{t \geq 0 \mid \text{ for all } [t,\ t+\delta),\ \delta > 0, \exists\, t_1, \\ t < t_1 < t+\delta, \text{ such that } y(t_1;\phi) \notin I\}\ . \quad (13)$$

First, assume that $y_\phi(t_0) = b$. In this case, for every $(t_0,\ t_0+\delta)$, $\delta > 0$, we can find a point $t_2, t_0 < t_2 < t_0 + \min(\delta, 1)$, such that $y_\phi(t_2) > b$ and $\dot{y}_\phi(t_2) > 0$. However, since $y_\phi(t_2 - \tau(t)) \leq \sup_{t_2 - d_{max} \leq s \leq t_2 - 1} y_\phi(s) \leq b$ from (13) and due to the fact that $1 \leq \tau(t) \leq d_{max}$, we have

$$\dot{y}_\phi(t_2) = \frac{\overline{\kappa}(y_\phi(t_2))\left(f(g^{-1}(y_\phi(t_2 - \tau(t)))) - y_\phi(t_2)\right)}{\nu} \\ < 0$$

from (9) and Assumption 3 that I is invariant under F, *i.e.,* $F(y_\phi(t_2 - \tau(t))) = f(g^{-1}(y_\phi(t_2 - \tau(t)))) \leq b$. This contradicts the earlier assumption that $\dot{y}_\phi(t_2) > 0$.

Similarly, suppose that $y_\phi(t_0) = a$ and the trajectory exits from the left end of the interval. Then, for every interval $(t_0,\ t_0+\delta)$, $\delta > 0$, we can find $t_2, t_0 < t_2 < t_0 + \min(\delta, 1)$, such that $0 < y_\phi(t_2) < a$ and $\dot{y}_\phi(t_2) < 0$. However, because $y_\phi(t_2 - \tau(t)) \geq \inf_{t_2 - d_{max} \leq s \leq t_2 - 1} y_\phi(s) \geq a$, we have

$$\dot{y}_\phi(t_2) = \frac{\overline{\kappa}(y_\phi(t_2))(f(g^{-1}(y_\phi(t_2 - \tau(t)))) - y_\phi(t_2))}{\nu} \\ > 0$$

from (9) and Assumption 3. This, however, contradicts the assumption $\dot{y}_\phi(t_2) < 0$. Hence, the theorem follows. ■

Next theorem considers the case when the map F has an attracting fixed point $y^*(= g(x^*))$ with immediate basin of attraction $J_0 : F^n y_0 \to y^*$ for any $y_0 \in J_0$, which is assumed to be a closed set invariant under F. Let $X_{J_0} = C([-d_{max}, 0], J_0)$. Then, the following theorem holds.

Theorem 2. (Stability) For any $\nu > 0$ and $\phi \in X_{J_0}$, we have $\lim_{t\to\infty} y_\phi(t) = y^*$.

Before proving the theorem we will first state a lemma which is the key to the proof of Theorem 2.

Lemma 1. Suppose that a closed interval J is mapped by F to itself. If neither of the endpoints of the interval $F(J)$ is a fixed point, then for every $\phi \in X_J = C([-d_{max}, 0], J)$ there exists a finite $t_0 = t_0(\phi, \nu, \kappa)$ such that $y_\phi(t) \in F(J)$ for all $t \geq t_0$.

Proof: From Theorem 1 it is clear that $y_\phi(t) \in J$ for all $t \geq 0$. The claim here is that after some time t_0 $y_\phi(t)$ will belong to $F(J) \subset J$.

We consider two cases. First, assume that $\phi(0) \in F(J)$. Then it can be shown that $y_\phi(t) \in F(J)$ for all $t \geq 0$ by contradiction. Suppose that this is not true, and let t_0 be the first time when $y_\phi(t)$ leaves the interval $F(J)$. In particular, assume that it leaves the interval through the right end, *i.e.,* every time interval $(t_0,\ t')$, where $t' > t_0$, contains a point t_1 such that $y_\phi(t_1) > \sup F(J)$. The interval $(t_0,\ t')$ also contains a point t_2 such that $y_\phi(t_2) > \sup F(J)$ and $\dot{y}_\phi(t_2) > 0$. However, as $y_\phi(t) \in J$ for all $t \in [t_0 - d_{max}, t_0]$, we have

$$\dot{y}_\phi(t_2) = \frac{\overline{\kappa}(y_\phi(t_2))(f(g^{-1}(y_\phi(t_2 - \tau(t)))) - y_\phi(t_2))}{\nu} \\ < 0$$

from (9). This contradicts the earlier assumption that $\dot{y}_\phi(t_2) > 0$. The other case where $y_\phi(t)$ leaves the interval from the left end can be handled similarly.

Now assume that $\phi(0) \notin F(J)$. In particular, let $\phi(0) > \sup F(J)$. We claim that $y_\phi(t)$ is decreasing for all $t \in [0, t_0]$, where $t_0 \leq \infty$ is the first time such that $y_\phi(t_0) = \sup F(J)$. We first argue that $t_0 < \infty$ by contradiction. Suppose that $y_\phi(t) > \sup F(J)$ for all $t \geq 0$. From (9), for all $t \geq 0$, we have

$$\dot{y}_\phi(t) = \frac{\overline{\kappa}(y_\phi(t))(f(g^{-1}(y_\phi(t - \tau(t)))) - y_\phi(t))}{\nu} \\ < 0$$

because $f(g^{-1}(y_\phi(t - \tau(t)))) \leq \sup F(J)$. Then, there exists a limit $\overline{y} = \lim_{t\to\infty} y_\phi(t) \geq \sup F(J)$ due to Bolzano-Weierstrass theorem (Fitzpatrick, 1996) which says that every strictly decreasing sequence which is bounded from below has a limit. As $\overline{y}$ is not a fixed point of map F, $\overline{\kappa}(\overline{y})(\overline{y} - f(g^{-1}(\overline{y}))) := \delta > 0$. This tells us from (9) that

$$\dot{y}_\phi(t) = \frac{\overline{\kappa}(y_\phi(t))(f(g^{-1}(y_\phi(t - \tau(t)))) - y_\phi(t))}{\nu} \\ < -\frac{\delta}{2\nu}$$

for all sufficiently large t. This implies that $y_\phi(t) \to -\infty$ as $t \to \infty$, which is a contradiction, because this means that $y_\phi(t)$ crosses $\sup F(J)$ for some finite t. Hence, $t_0 < \infty$. Now, we can apply the argument used in the first case, *i.e.,* $\phi(0) \in F(J)$, to the system with $y(t_0) = \sup F(J)$ and $y(t) \in J$ for all $t \in [t_0 - d_{max},\ t_0]$ to prove that $y(t) \in F(J)$ for all $t \geq t_0$. The other case where $\phi(0) < \inf F(J)$ can be handled similarly. ■

Now we provide the proof for Theorem 2 using Lemma 1.

Proof: (Theorem 2) For any $\phi \in X_{J_0}$, define $m = \inf\{\phi(s),\ s \in [-d_{max},\ 0]\}$ and $M = \sup\{\phi(s),\ s \in [-d_{max},\ 0]\}$. Clearly, $[m,\ M] \subset J_0$. Let $J^{'}$ be the smallest closed invariant interval containing $[m,\ M]$ which is a subset of J_0. The existence of such an interval is guaranteed from the assumption that J_0 is invariant under F. Then, from the existence of a stable fixed point of the map F, $J^{'} \supset F(J^{'}) \supset F(F(J^{'})) \supset \ldots$ and $\cap_{i \geq 0} F^i(J^{'}) = \{y^*\}$. Using the invariance result and Lemma 1 repeatedly one can find a sequence of $t_i, i \geq 1$, such that for all $t \geq t_i$, $y_\phi(t) \in F^i(J^{'})$. The convergence now follows from the assumption that $F^i(J^{'}) \rightarrow \{y^*\}$. ■

The above theorem tells us that if the initial function lies in X_{J_0}, then the rate $x(t)$ converges to the solution of (1) regardless of the value of T, κ, or exact dynamical nature of queuing delay. Hence, it establishes a strong convergence result in the presence of a time-varying communication delay without any conditions on the value of T or κ. This contrasts the results obtained in (Deb and Srikant, 2002; Mazenc and Niculescu, 2003).

3. CONCLUSION

We have considered the stability of a rate control system in a simple single-flow, single-resource context in the presence of a state-dependent time-varying delay. We have shown that the stability conditions for a fixed delay are sufficient even with a time-varying delay.

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