Convergence results for ant routing

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***Abstract*— We study the convergence property of a family of distributed routing algorithms based on the ant colony metaphor, namely the uniform and regular ant routing algorithms discussed by Subramanian et al. [8]. For a simple two-node network, we show that the probabilistic routing tables converge in distribution (resp. in the a.s. sense) for the uniform (resp. regular) case. To the best of the authors' knowledge, the results given here appear to be the first formal convergence results for ant routing algorithms. Although they hold only for a very limited class of networks, their analysis already provides some useful lessons for extending the results to more complicated networks. We also discuss implementation issues that naturally arise from the convergence analysis.**

I. Introduction

In the past decade, several authors have used the ant colony metaphor to design distributed adaptive routing algorithms in both datagram networks [5] and telephone networks [7]; a good survey of these efforts is given in Chapter 2 of [1]. More recently, these ideas have been proposed in the context of mobile ad-hoc networks (MANETs), e.g., [6].

A common feature shared by these algorithms is their attempt to reproduce the collective behavior of a swarm of insects to solve a relatively complex problem in a distributed manner, typically using only local information. This new paradigm has the potential to deliver a new way of designing robust, scalable, adaptive, and distributed algorithms for network resource management.

Although many ant colony optimization (ACO)-based algorithms [1] have been proposed and studied in the past, little is known about their convergence properties. There are several difficulties in establishing such convergence results. First, it is not always easy to identify a suitable mathematical framework for analyzing these algorithms at some level of generality, thereby perhaps explaining the lack of a "general theory" of ant (routing) algorithms. Secondly, even when such a framework is available, many state variables which are required to capture the dynamics of the algorithms, are coupled in their evolution. Often this coupling severely limits the usefulness of traditional techniques of convergence analysis.

In this paper we are concerned specifically with the convergence properties of the class of *ant routing* algorithms discussed by Subramanian et al. [8]. These algorithms are randomized algorithms that implement a form of *backward* learning in response to short control messages called *ants*.

A. Ant routing

Consider the situation where hosts are provided connectivity through a network of R routers. Two routers are said to be neighbors if there exists a bidirectional point-to-point link between them, and let $\mathcal{N}_r$ denote the set of routers which are neighbors of router r ($r = 1, \ldots, R$). Router r maintains a probabilistic routing table with a separate vector entry $(d, (i, p_i),\ i \in \mathcal{N}_r)$ for each host destination d. For each neighboring router i in $\mathcal{N}_r$, we understand p_i as the probability with which router r uses link (r, i) when forwarding a *data* packet destined for d. There is a cost c_{ri} associated with the use of link (r, i); this cost is assumed symmetric (i.e., $c_{ri} = c_{ir}$) and is known to router r.

An ant is a control message of the form $(d, s \| c)$ where d and s are distinct hosts and c is some numerical value to be updated in due course. We refer to hosts d and s as the destination and source, respectively, and regard c as an estimate of the cost-to-go for reaching host d. Periodically, host d generates an ant $(d, s \| c)$ which is destined for some randomly selected host $s \neq d$ with c initially set to zero. The ant is forwarded to the source host s over the network of interconnected routers and on the way updates the routing tables at intermediary routers (in a way to specified shortly).

When ant $(d, s \| c)$ arrives at the intermediary router r coming from router i through link (i, r), the cost estimate c for reaching d from router i is incremented by the cost of the (reverse) link (r, i) with

$$c \leftarrow c + c_{ri}, \tag{1}$$

and this new value of c thus provides an estimate of the cost-to-go to reach d from router r. Next, the vector entry in the routing table for destination d is updated according to

$$p_i \leftarrow \frac{p_i + \Delta}{1 + \Delta}, \quad p_j \leftarrow \frac{p_j}{1 + \Delta} \ \ (j \in \mathcal{N}_r \setminus \{i\}) \tag{2}$$

where $\Delta = f(c)$ and $f(c)$ is a *decreasing* function of the just incremented value of c. Consequently, the probability of using the (reverse) link over which the ant arrived at router r has been increased relative to that of other links, while the probability of using the other links is discounted.

Upon completing these various updates, router r forwards the updated ant $(d, s \| c)$ to one of its neighboring router i in $\mathcal{N}_r$. The ant $(d, s \| c)$ eventually reaches its destination s with c now giving the end-to-end cost value of sending a message from s to d, and is destroyed. In fact, the final value of c is simply the cost of traversing the network from s to d along the (reverse) path followed by the ant.

B. Convergence results

The manner in which ant forwarding is carried out distinguishes the various types of ant routing algorithms studied by Subramanian et al. [8]. This reference deals mostly with a simple two-router network, announces some convergence results but provides no *formal* proofs. To the best of the authors' knowledge, this is the extent of the current state of knowledge concerning the *mathematical* convergence of these ant algorithms. In this paper we take some steps towards remedying this state of affairs. Below we review the convergence statements of [8] and [9], and present our results. Proofs are outlined in the body of the paper, with additional details available in [10].

1. *Uniform* ant algorithms are designed with *multi-path* routing in mind, and require that ants arriving, say at router r, be forwarded to the next neighboring router with equal probability L_r^{-1} where L_r is the number of routers in $\mathcal{N}_r$.[1] As a result, uniform ants will utilize each and every path between a source and a destination with a positive probability. It is claimed [8, Prop. 2] [9] that the routing tables converge (in an unspecified mode of convergence) to constant values. Here, for the two-node network, we show that (i) convergence takes place in *distribution*, and (ii) *not* to constants. This last fact has implications for the implementation of such ant algorithms [Section V].

The key observation behind the convergence result for uniform ants is the identification of the iterates of the uniform ant algorithm as the output of a very simple collection of iterated random functions. A large literature is available on the convergence of these iterative schemes, and the survey in [4] (and references therein) provides a nice introduction to this topic. We exploit the very simple structure of the underlying collection of random functions to give a simple and self-contained proof.

2. On the other hand, *regular* ant algorithms attempt to find the least cost (or shortest) path(s). This is achieved by forwarding the ants according to the probabilistic routing tables used to implement the routing of data packets. For the two-node network the algorithm is claimed [8, Prop. 1] [9] to converge (presumably in the a.s. mode of convergence) to constant values, but no proof is provided in that reference. Here, in the case of *synchronous* instantaneous updates, we show by martingale techniques that regular ant algorithm will indeed lead asymptotically to least path discovery, in the two-node network, as desired!

True, while the results given here are perhaps the first formal convergence results for ant routing, they hold only for a very limited class of networks, i.e., a two-node network. Yet, the underlying analyses given below already provide some useful lessons for extending the results to more complicated networks. This is discussed at the end of Sections VI and VII, respectively.

The paper is organized as follows: Ant algorithms are formally described in Section II for a simple two-link network. (Generalized) uniform ant and regular ant algorithms are introduced in Sections III and IV, respectively. The main convergence results are given in Theorems 1 and 2 which are established in Sections VI and VII, respectively. Some implementation issues are pointed out in Section V.

[1]More generally, L_r is the number of links going out of r in the case of multiple links.

A word on the notation: For any integer L, the ℓ^{th} component of any element $\boldsymbol{x}$ in $\mathbb{R}^L$ is denoted by x_ℓ, $\ell = 1, \ldots, L$, so that $\boldsymbol{x} \equiv (x_1, \ldots, x_L)$. A similar convention is used for $\mathbb{R}^L$-valued random variables (rvs).

II. The two-node model

The network is composed of two routers or nodes, thereafter labeled node $i = 0$ and node $i = 1$, connected by a set of L parallel bidirectional links.[2] All hosts are attached to either node $i = 0$ or node $i = 1$. Each link $\ell = 1, \cdots, L$, has a transmission cost of $c_\ell > 0$ in either direction. Without loss of generality we assume the links to be labeled in order of increasing cost with

$$c_1 \leq c_2 \leq \cdots \leq c_L. \tag{3}$$

Pick node i ($i = 0, 1$). Destination hosts attached to node i generate ants at times $\{t_n^i,\ n = 1, 2, \ldots\}$ with $t_n^i < t_{n+1}^i$ for each $n = 1, 2, \ldots$. Forwarding the n^{th} ant to node $1 - i$ requires that one of the L links from node i to node $1 - i$, say $\ell_{1-i}(n)$, be selected. The n^{th} ant is then sent over link $\ell_{1-i}(n)$, and arrives at node $1-i$, say at time a_n^i with $t_n^i \leq a_n^i$. For simplicity of exposition we assume the non-overtaking condition

$$a_n^i < a_{n+1}^i, \quad n = 1, 2, \ldots \tag{4}$$

and for convenience we take a_0^i arbitrary such that $a_0^i \leq a_1^i$.

Node i maintains a probabilistic forwarding table

$$\boldsymbol{p}^i(n) := (p_1^i(n), \cdots, p_L^i(n)) \tag{5}$$

with $0 \leq p_\ell^i(n) \leq 1$ ($\ell = 1, \ldots, L$) and $\sum_{\ell=1}^L p_\ell^i(n) = 1$. The entry $p_\ell^i(n)$ is interpreted as the probability that during the interval $[a_n^{1-i}, a_{n+1}^{1-i})$, node i forwards a data packet from node i to node $1 - i$ over link ℓ. Its precise meaning will be clarified in due time.

When, at time a_{n+1}^i, node $1 - i$ receives the $(n+1)^{st}$ ant, say over link $\ell_{i-1}(n + 1) = \ell$, it immediately updates its probabilistic forwarding table according to

$$p_\ell^{1-i}(n+1) = \frac{p_\ell^{1-i}(n) + C_\ell(1-i)}{1 + C_\ell(1-i)} \tag{6}$$

and

$$p_k^{1-i}(n+1) = \frac{p_k^{1-i}(n)}{1 + C_\ell(1-i)}, \quad k \neq \ell,\ k = 1, \ldots, L \tag{7}$$

for constants $C_\ell(1 - i) > 0$. The selection of these constants is discussed later. The cost update (1) is superfluous here due to the simplified structure of this two-node network.

To completely specify ant algorithms we need to provide the rule by which the links $\{\ell_0(n), \ell_1(n),\ n = 1, 2, \ldots\}$ are selected. This will be done in Sections III and IV for uniform and regular ant routing, respectively.

[2]These links can be also thought of as disjoint paths.

Before doing so, we introduce some notation: For each $i = 0, 1$, and for each link $\ell = 1, \cdots, L$, define the mapping $\phi^i_\ell : [0,1]^L \to [0,1]^L$ by

$$\phi^i_{\ell,k}(\boldsymbol{p}) := \begin{cases} \frac{p_\ell + C_\ell(i)}{1+C_\ell(i)} & k = \ell \\ \frac{p_k}{1+C_\ell(i)} & k \neq \ell \end{cases}, \quad \boldsymbol{p} \in [0,1]^L \tag{8}$$

for the constants $C_\ell(i) > 0$ appearing in (6) and (7). These updating rules can now be written more compactly as

$$\boldsymbol{p}^i(n+1) = \phi^i_\ell(\boldsymbol{p}^i(n)) \quad \text{if} \quad \ell_i(n+1) = \ell, \quad n = 0, 1, \ldots \tag{9}$$

with $\boldsymbol{p}^i(0)$ denoting the forwarding probability vector initially stored at node i (i.e., at time a^i_0).

III. Generalized uniform ants

The class of ant algorithms we now introduce is somewhat more general than the one discussed in [8]. We specify how ants are propagated by assuming that the $\{1, \ldots, L\}$-valued rvs $\{\ell_0(n), \ell_1(n),\ n = 1, 2, \ldots\}$ are *mutually independent* rvs which are taken to be *independent* of the initial conditions $\boldsymbol{p}^0(0)$ and $\boldsymbol{p}^1(0)$. Moreover, for each $i = 0, 1$, the rvs $\{\ell_i(n),\ n = 1, 2, \ldots\}$ are taken to be *i.i.d.* rvs distributed according to some probability mass function (pmf) $\boldsymbol{v}^i = (v^i_1, \ldots, v^i_L)$ on $\{1, \ldots, L\}$.

Under these assumptions, it is plain from (6) and (7) that the probabilistic forwarding tables of the nodes are updated independently of each other, whence the evolutions of the tables $\{\boldsymbol{p}^0(n),\ n = 0, 1, \ldots\}$ and $\{\boldsymbol{p}^1(n),\ n = 0, 1, \ldots\}$ are decoupled. Their long-term behavior is summarized in the next result where $\Longrightarrow_n$ denotes convergence in distribution (with n going to infinity).

Theorem 1: Under the foregoing assumptions we have:

(i) For each $i = 0, 1$, the limit

$$\boldsymbol{p}^i_\star = \lim_{n\to\infty} \left(\phi^i_{\ell_i(1)} \circ \phi^i_{\ell_i(2)} \circ \ldots \circ \phi^i_{\ell_i(n)} \right)(\boldsymbol{p}) \tag{10}$$

exists for each $\boldsymbol{p}$ in $[0,1]^L$, and is independent of $\boldsymbol{p}$;

(ii) Moreover, there exists a pair of independent $[0,1]^L$-valued rvs $\boldsymbol{p}^0$ and $\boldsymbol{p}^1$ distributed like $\boldsymbol{p}^0_\star$ and $\boldsymbol{p}^1_\star$, respectively, such that

$$(\boldsymbol{p}^0(n), \boldsymbol{p}^1(n)) \Longrightarrow_n (\boldsymbol{p}^0, \boldsymbol{p}^1) \tag{11}$$

regardless of the initial condition $(\boldsymbol{p}^0(0), \boldsymbol{p}^1(0))$.

The proof of Theorem 1 is given in Section VI. We emphasize that the result holds for a class of algorithms which is somewhat more general than the one introduced in [8]: Indeed, the positive constants entering (6)-(7) are arbitrary and need not be constrained to

$$C_\ell(i) = f(c_\ell) =: \Delta_\ell \quad i = 0, 1,\ \ell = 1, \ldots, L \tag{12}$$

for some strictly decreasing function $f : \mathbb{R} \to (0, \infty)$. Next, when we take the pmfs $\boldsymbol{v}^0$ and $\boldsymbol{v}^1$ to be the uniform pmf on $\{1, \ldots, L\}$, we recover the case discussed in [8], hence the name uniform ant algorithm.

Finally, the assumptions enforced on the "reception" times $\{a^i_n,\ n = 1, 2, \ldots\}$ $(i = 1, 0)$ can be weakened considerably: These times could in principle be *random* and Theorem 1 would still hold. The non-overtaking assumption (4) can also be dropped if we now interpret $\ell_{1-i}(n)$ as the identity of the link from node i to node $1-i$ which was traversed by the n^{th} ant *received* at node $1-i$ at the (possibly random) time a^i_n. Then, under the aforementioned independence assumptions, Theorem 1 will still hold.

IV. Regular ants with synchronous updates

With regular ant algorithms, the ants are forwarded according to the routing table at the router, and the update processes at the two nodes are now coupled. This renders the analysis of regular ant algorithms more delicate than that of uniform ant algorithms. To make progress, we need assumptions on the *relative timing* of the various events. Here we consider the *ideal* situation where both nodes generate the ants in a synchronized manner and the updates take place simultaneously and instantaneously. Formally, we require

$$t^0_n = a^0_n = t^1_n = a^1_n, \quad n = 1, 2, \ldots \tag{13}$$

For each $n = 0, 1, \ldots$, we introduce the σ-field $\mathcal{F}_n$ generated by the rvs

$$\{\boldsymbol{p}^0(0), \boldsymbol{p}^1(0), \boldsymbol{p}^0(k), \boldsymbol{p}^1(k), \ell_0(k), \ell_1(k),\ k = 1, 2, \ldots, n\}.$$

The rvs $\ell_0(n+1)$ and $\ell_1(n+1)$ are then specified to be conditionally independent given $\mathcal{F}_n$ with

$$\mathbf{P}\left[\ell_i(n+1) = \ell | \mathcal{F}_n\right] = p^{1-i}_\ell(n) \tag{14}$$

for each $\ell = 1, \ldots, L$ and $i = 0, 1$. The prescription (14) encodes the fact that ants are routed through the network according to the very routing tables they help update.

The constants entering (6)-(7) will be assumed of the form (12). The entry for link ℓ at node $1-i$ is increased proportionally to $f(c_\ell)$, which is a *strictly* decreasing function of c_ℓ, while those of the other links are discounted by $(1+f(c_\ell))^{-1}$.

Combining (14) with (6)-(7), we can write the evolution of the routing tables in the more suggestive (yet somewhat sloppy) manner: Fix $i = 0, 1$. For each $\ell = 1, \ldots, L$, we have

$$p^i_\ell(n+1) = \begin{cases} \frac{p^i_\ell(n) + f(c_\ell)}{1+f(c_\ell)} & \text{w.p. } p^{1-i}_\ell(n) \\ \\ \frac{p^i_\ell(n)}{1+f(c_j)} & \text{w.p. } p^{1-i}_j(n),\ j \neq \ell \end{cases} \tag{15}$$

for all $n = 0, 1, \ldots$. This formulation clearly shows the coupling of the entries $\boldsymbol{p}^0(n)$ and $\boldsymbol{p}^1(n)$ in the routing tables.

Writing

$$\boldsymbol{p}(n) = (\boldsymbol{p}^0(n), \boldsymbol{p}^1(n)), \quad n = 0, 1, \ldots$$

we readily see from (9) and (14) that the sequence of successive routing tables $\{\boldsymbol{p}(n),\ n = 0, 1, \ldots\}$ form a time-homogeneous Markov chain on the non-countable state space $[0,1]^L \times [0,1]^L$. A natural question is whether the Markov chain converges to the desired operating point, i.e., $((1, 0, \cdots, 0), (1, 0, \cdots, 0))$, which corresponds to the least-cost path. This is the content of the next result.

Theorem 2: Assume f to be strictly decreasing and that $c_1 < c_\ell$ for all $\ell = 2, \cdots, L$. For each $i = 0, 1$, the routing tables $\{\boldsymbol{p}^i(n),\ n = 0, 1, \ldots\}$ converge a.s. with

$$\lim_{n\to\infty} \boldsymbol{p}^i(n) = (1, 0, \cdots, 0) \quad a.s.$$

provided the initial conditions are selected such that

$$p_1^0(0) + p_1^1(0) > 0. \tag{16}$$

Under the enforced assumptions, there is only one link with minimal cost and it is link 1.

V. Implementation issues

We briefly discuss some of the implementation issues associated with the ant routing algorithms discussed earlier. For simplicity, we do so for the case first described in [8] with $L = 2$ where the constants in the probability updates (6) and (7) are selected according to (12).

A. Uniform ant routing

Take the pmfs $\boldsymbol{v}^0$ and $\boldsymbol{v}^1$ to be the uniform pmf on $\{1, 2\}$. It is easy to see that the probabilistic routing tables evolve according to

$$\boldsymbol{p}^i(n+1) = \begin{cases} \left[\frac{p_1^i(n)+\Delta_1}{1+\Delta_1}, \frac{p_2^i(n)}{1+\Delta_1}\right] & \text{w.p. } \frac{1}{2} \\ \\ \left[\frac{p_1^i(n)}{1+\Delta_2}, \frac{p_2^i(n)+\Delta_2}{1+\Delta_2}\right] & \text{w.p. } \frac{1}{2} \end{cases} \tag{17}$$

where the constants Δ_ℓ ($\ell = 1, 2$) are given by (12).

Selecting the proper values of $f(c_\ell)$ ($\ell = 1, 2$) is crucial here for good performance. Indeed, if the parameters are selected such that

$$\frac{1}{1+\Delta_2} < \frac{\Delta_1}{1+\Delta_1},$$

then the iterates produced by (17) exhibit an oscillatory behavior with successive values possibly bouncing around between the *non-overlapping* intervals $(0, (1+\Delta_2)^{-1})$ and $((1+\Delta_1)^{-1}\Delta_1, 1)$. This behavior leads to undesirable oscillations in the routing tables, and is clear evidence that the convergence of Theorem 1 cannot be in the a.s. sense. In fact, convergence takes place in distribution (*not* a.s.) to a limiting random variable whose distribution has a *non-connected* support on the interval $[0, 1]$.

Subramanian et al. [8, Prop. 2] claim the convergence

$$\lim_{n\to\infty} \boldsymbol{p}^i(n) = L_i, \quad i = 0, 1$$

for some *constants* L_0 and L_1, without further indication of the mode of convergence used for this convergence statement which involve rvs. As the previous discussion implies, this cannot be correct.

B. Regular ant routing

By Theorem 2 the regular ant routing algorithm converges to the desired operating point, namely the least-cost link. However, if the algorithm is implemented on a machine with *finite* precision, as is always the case with computer implementations, there is a positive probability that the algorithm will converge to the path with a larger cost. Indeed, suppose that the product space $[0, 1] \times [0, 1]$ is approximated by a two-dimensional grid with a finite number of points, say $(\frac{k}{K}, \frac{\ell}{K})$ (with $k, \ell = 0, \ldots, K$). We can then approximate $(p_1^0(n), p_1^1(n))$ using the closest point in the grid, with resulting output $(\tilde{p}_1^0(n), \tilde{p}_1^1(n))$ evolving according to a time-homogeneous *finite-state* Markov chain described by projecting the right hand sides of (15) onto the grid. Regardless of the value of K, the corner point $(0, 0)$ is now an absorbing state, and the probability of reaching it is strictly positive, so that the probability of converging to the shortest path is strictly smaller than one.

VI. A proof of Theorem 1

As remarked earlier, because the forwarding probability tables evolve independently of each other, we need only establish the convergence $\boldsymbol{p}^i(n) \Longrightarrow_n \boldsymbol{p}_\star^i$ for each $i = 0, 1$.

Thus fix $i = 0, 1$ and pick $n = 0, 1, \ldots$. Iterating yields the relation

$$\begin{aligned} & \boldsymbol{p}^i(n+1) \\ = & \left(\phi^i_{\ell_i(n+1)} \circ \phi^i_{\ell_i(n)} \circ \ldots \circ \circ \phi^i_{\ell_i(2)} \circ \phi^i_{\ell_i(1)}\right)(\boldsymbol{p}^i(0)). \end{aligned} \tag{18}$$

The key observation is the stochastic equivalence [3]

$$\begin{aligned} & \boldsymbol{p}^i(n+1) \\ =_{st} & \left(\phi^i_{\ell_i(1)} \circ \phi^i_{\ell_i(2)} \circ \ldots \circ \phi^i_{\ell_i(n)} \circ \phi^i_{\ell_i(n+1)}\right)(\boldsymbol{p}^0(0)) \end{aligned} \tag{19}$$

which holds by virtue of the i.i.d. assumption on the sequence $\{\ell_i(n),\ n = 1, 2, \ldots\}$ and its independence from $\boldsymbol{p}^i(0)$. The convergence $\boldsymbol{p}^i(n) \Longrightarrow_n \boldsymbol{p}_\star^i$ will follow if we can establish the pointwise convergence

$$\lim_{n\to\infty} \left(\phi^i_{\ell_i(1)} \circ \phi^i_{\ell_i(2)} \circ \ldots \circ \phi^i_{\ell_i(n)}\right)(\boldsymbol{p}^0(0)) = \boldsymbol{p}_\star^i \tag{20}$$

with $\boldsymbol{p}_\star^i$ defined through the *pointwise* convergence (10).

To establish (10) with $\boldsymbol{p}_\star^i$ independent of the initial condition, we proceed as follows: Equip $\mathbb{R}^L$ with the norm defined by $\|\boldsymbol{x}\| := \sum_{\ell=1}^L |x_\ell|$ for any vector $\boldsymbol{x}$ in $\mathbb{R}^L$. This norm is equivalent to the usual Euclidean norm, but easier to use here.

[3] Two $\mathbb{R}$–valued rvs X and Y are said to be equal in law if they have the same distribution, a fact we denote by $X =_{st} Y$.

Fix $\ell = 1, \ldots, L$. For arbitrary $\boldsymbol{x}$ and $\boldsymbol{y}$ in $[0,1]^L$, we get

$$\begin{aligned}
& \|\phi^i_\ell(\boldsymbol{x}) - \phi^i_\ell(\boldsymbol{y})\| \\
= & \sum_{k=1}^{L} \left|\phi^i_{\ell,k}(\boldsymbol{x}) - \phi^i_{\ell,k}(\boldsymbol{y})\right| \\
= & \sum_{k\neq\ell} \left|\frac{x_k - y_k}{1+C_\ell(i)}\right| + \frac{|x_\ell + C_\ell(i) - (y_\ell + C_\ell(i))|}{1+C_\ell(i)} \\
= & \frac{\|\boldsymbol{x}-\boldsymbol{y}\|}{1+C_\ell(i)} \\
\leq & K_i\|\boldsymbol{x}-\boldsymbol{y}\|
\end{aligned}$$

with

$$K_i := \max_{\ell=1,\ldots,L} (1+C_\ell(i))^{-1} < 1.$$

Therefore,

$$\max_{\ell=1,\ldots,L} \|\phi^i_\ell(\boldsymbol{x}) - \phi^i_\ell(\boldsymbol{y})\| \leq K_i\|\boldsymbol{x}-\boldsymbol{y}\| \ . \tag{21}$$

Using this last inequality, we check (by induction on $n = 1,2,\ldots$) that

$$\begin{aligned}
& \| \left(\phi^i_{\ell_i(1)} \circ \ldots \circ \phi^i_{\ell_i(n-1)} \circ \phi^i_{\ell_i(n)}\right)(\boldsymbol{p}) \\
& \quad - \left(\phi^i_{\ell_i(1)} \circ \ldots \circ \phi^i_{\ell_i(n-1)} \circ \phi^i_{\ell_i(n)}\right)(\boldsymbol{q})\| \\
\leq & K_i^n \|\boldsymbol{p}-\boldsymbol{q}\|
\end{aligned} \tag{22}$$

for arbitrary $\boldsymbol{p}$ and $\boldsymbol{q}$ in $[0,1]^L$. Furthermore, for each $m = 1,2,\ldots$, we get

$$\begin{aligned}
& \| \left(\phi^i_{\ell_i(1)} \circ \phi^i_{\ell_i(2)} \circ \ldots \circ \phi^i_{\ell_i(n+m)}\right)(\boldsymbol{p}) \\
& \quad - \left(\phi^i_{\ell_i(1)} \circ \phi^i_{\ell_i(2)} \circ \ldots \circ \phi^i_{\ell_i(n)}\right)(\boldsymbol{p})\| \\
\leq & K_i^n \| \left(\phi^i_{\ell_i(n+1)} \circ \phi^i_{\ell_i(n+2)} \circ \ldots \circ \phi^i_{\ell_i(n+m)}\right)(\boldsymbol{p}) - \boldsymbol{p}\| \\
\leq & K_i^n L \quad \text{uniformly in } m.
\end{aligned} \tag{23}$$

By the fact $K_i < 1$, it follows for each $\boldsymbol{p}$ that the sequence

$$\{\left(\phi^i_{\ell_i(1)} \circ \phi^i_{\ell_i(1)} \circ \ldots \circ \phi^i_{\ell_i(n)}\right)(\boldsymbol{p}), \quad n = 1,2,\ldots\}$$

forms a Cauchy sequence, hence is convergent. Eqn. (22) shows that the limit of this convergent sequence is independent of $\boldsymbol{p}$. This establishes (10) and the proof is now complete. ■

A careful examination of the arguments given in the proof of Theorem 1 points to the possibility of establishing the convergence of uniform ants for more general network topologies. Indeed, this is due to the fact that in uniform ant routing, the evolution of the routing tables are decoupled. It is also clear that weaker assumptions can be imposed on the link selection rvs, e.g., the proof given above goes through if each of the sequences $\{\ell_i(n),\ n = 1,2,\ldots\}$ $(i = 1,0)$ is assumed stationary and reversible.

VII. A PROOF OF THEOREM 2

We need to show that each of the sequences $\{\boldsymbol{p}^0(n),\ n = 0,1,\ldots\}$ and $\{\boldsymbol{p}^1(n),\ n = 0,1,\ldots\}$ converges a.s. to the vector $(1,0,\cdots,0)$. For each $i = 0,1$, this is equivalent to showing that

$$\lim_{n\to\infty} p^i_1(n) = 1 \quad a.s. \tag{24}$$

since $\sum_{\ell=1}^L p^i_\ell(n) = 1$ for all $n = 0,1,\ldots$.

To establish (24), we introduce the rvs $\{Z(n),\ n = 0,1,\ldots\}$ given by

$$Z(n) = p^0_1(n) + p^1_1(n), \quad n = 0,1,\ldots \tag{25}$$

and note that the convergence (24) for both $i = 0$ and $i = 1$ will follow if we can show that

$$\lim_{n\to\infty} Z(n) = 2 \quad a.s. \tag{26}$$

To do so, we first note that $f(c_1) > f(c_\ell)$ for each $\ell = 2,\cdots,L$ under the enforced assumptions on f and on the link costs. Fix $i = 0,1$ and $n = 0,1,\ldots$. Combining (9) and (14), we find

$$\mathbf{E}\left[p^i_k(n+1)|\mathcal{F}_n\right] = \sum_{\ell=1}^L p^{1-i}_\ell(n)\phi^i_{\ell,k}(p^i(n)) \quad a.s. \tag{27}$$

for each $k = 1,\ldots,L$. Next, reporting (27) into the definition (25), and using the constants (12) in (8), we readily conclude that

$$\begin{aligned}
& \mathbf{E}\left[Z(n+1) \mid \mathcal{F}_n\right] - Z(n) \\
= & \sum_{\ell=2}^L \frac{f(c_1)-f(c_\ell)}{(1+f(c_1))(1+f(c_\ell))}\left(p^0_1(n)p^1_\ell(n) + p^1_1(n)p^0_\ell(n)\right)
\end{aligned} \tag{28}$$

Consequently,

$$\mathbf{E}\left[Z(n+1) \mid \mathcal{F}_n\right] \geq Z(n) \quad a.s. \tag{29}$$

and the rvs $\{Z(n),\ n = 0,1,2,\ldots\}$ form a bounded $\mathcal{F}_n$-submartingale with $0 \leq \mathbf{E}\left[Z(n)\right] \leq 2$, for all $n = 0,1,\ldots$. By the Martingale Convergence Theorem [3, Thm. 9.4., p. 334], the submartingale $\{Z(n),\ n = 0,1,2,\ldots\}$ converges a.s. to some rv Z, and we must have $0 \leq Z \leq 2$.

We note from (28) that the a.s. inequality in (29) becomes an equality if and only if

$$p^0_1(n)p^1_\ell(n) = p^1_1(n)p^0_\ell(n) = 0, \quad \ell = 2,\ldots,L$$

or equivalently, if and only if

$$p^0_1(n)(1-p^1_1(n)) = p^1_1(n)(1-p^0_1(n)) = 0.$$

In other words, the equalities hold if and only if either $p^0_1(n) = p^1_1(n) = 0$ or $p^0_1(n) = p^1_1(n) = 1$.

It is now a simple matter to check the following (say by direct inspection of the dynamics (15) for the routing tables): If $p^0_1(0) = p^1_1(0) = 0$, then $Z(n) = 0$ for all $n = 0,1,\ldots$ and $Z = 0$, while if $p^0_1(0) = p^1_1(0) = 1$, then $Z(n) = 2$ for all $n = 0,1,\ldots$ and $Z = 2$. On the other hand, if the initial conditions are selected so that $0 < Z(0) < 2$, then it is easy to see by induction that $0 < Z(n) < 2$ a.s. for all $n = 1,2,\ldots$.

To conclude the proof we need to show that $Z=2$ a.s. here as well. Proceeding by contradiction, we assume $\mathbf{P}[Z=2]<1$, in which case there exists some $\varepsilon>0$ such that

$$\delta := \mathbf{P}[\varepsilon < Z < 2-\varepsilon] > 0 \tag{30}$$

since $Z>0$ a.s. By the a.s. convergence of $\{Z(n),\ n=0,1,\ldots\}$ to Z we conclude that

$$\mathbf{P}\left[\frac{\varepsilon}{2} < Z(n) < 2-\frac{\varepsilon}{2}\right] \geq \frac{\delta}{2}, \quad n \geq n^\star \tag{31}$$

with $n^\star$ determined by ε and δ.

Write

$$K := \frac{f(c_1)-f(c_2)}{(1+f(c_1)^2)}.$$

Taking advantage of (28), for each $n=0,1,\ldots$, we get

$$\begin{aligned} & \mathbf{E}[Z(n+1)] - \mathbf{E}[Z(n)] \\ = \ & \mathbf{E}[\mathbf{E}[Z(n+1) \mid \mathcal{F}_n] - Z(n)] \\ = \ & \sum_{\ell=2}^{L} \frac{f(c_1)-f(c_\ell)}{(1+f(c_1))(1+f(c_\ell))} \cdot \mathbf{E}[\cdots] \\ \geq \ & K \cdot \mathbf{E}\left[\sum_{\ell=2}^{L} \left(p_1^0(n)p_\ell^1(n) + p_1^1(n)p_\ell^0(n)\right)\right] \\ = \ & K \cdot \mathbf{E}\left[p_1^0(n)(1-p_1^1(n)) + p_1^1(n)(1-p_1^0(n))\right] \\ \geq \ & K \cdot \mathbf{E}\left[(\cdots)\,\mathbf{1}\left[\frac{\varepsilon}{2} < Z(n) < 2-\frac{\varepsilon}{2}\right]\right] \end{aligned}$$

Next, with $K_\varepsilon := \{(x,y) \in [0,1]^2 : \frac{\varepsilon}{2} < x+y < 2-\frac{\varepsilon}{2}\}$, we note that

$$I(\varepsilon) := \inf\{x+y-2xy : \ (x,y) \in K_\varepsilon\} > 0.$$

Therefore, whenever $n \geq n^\star$, we find that

$$\begin{aligned} & \mathbf{E}[Z(n+1)] - \mathbf{E}[Z(n)] \\ \geq \ & KI(\varepsilon) \cdot \mathbf{P}\left[\frac{\varepsilon}{2} < Z(n) < 2-\frac{\varepsilon}{2}\right] \\ \geq \ & KI(\varepsilon)\frac{\delta}{2}, \end{aligned}$$

As a result, $\liminf_{n\to\infty} (\mathbf{E}[Z(n+1)] - \mathbf{E}[Z(n)]) > 0$, in contradiction with the convergence $\lim_{n\to\infty} \mathbf{E}[Z(n)] = \mathbf{E}[Z]$ that follows from the a.s. convergence of $\{Z(n),\ n=0,1,\ldots\}$ and the Bounded Convergence Theorem. The proof of (26) is now complete. ■

The convergence result of Theorem 2 cannot be obtained by traditional methods from the theory of Markov chains on general state spaces when applied to the time-homogeneous Markov chain $\{\boldsymbol{p}(n),\ n=0,1,\ldots\}$ on the non-countable state space $[0,1]^L \times [0,1]^L$. Indeed, the general theory identifies conditions akin to irreducibility which ensure the weak convergence (11). However, such irreducibility conditions typically imply a non-degenerate stationary distribution in the limit, at variance with the situation obtained here[4]!

[4]That is, if we eliminate the initial condition $Z(0)=0$.

The analysis above already reveals some of the difficulties inherent in establishing convergence results for regular ants. Our ability to establish extensions of Theorem 2 to more complex topologies is likely to depend on a careful timing of updating events. In that vein, we note that a version of Theorem 2 can also be obtained (through identical arguments) when the updates are triggered in an asynchronous manner according to mutually independent Poisson processes; we omit the details in the interest of brevity.

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